Lecture 26

Optical Theorem and Unstable Particles

We conclude this part of the course by considering unstable particles. These decay due to their interactions with other states into channels allowed by kinematics. We can understand their decay as a "leakage" effect on their propagation. In fact, as we will see below, their decay rate is related to the imaginary part of the two-point function. We discussed these points loosely when we introduced the decay rate of an unstable particle earlier. We will now derive an expression for the propagator of an unstable particle by showing that the imaginary part of the two-point function comes from particles on a loop diagram going on shell. For this we need to discuss the optical theorem first.

26.1 Optical Theorem

As we saw earlier, the **S** matrix can be written as

$$\mathbf{S} = \mathbf{1} + i\mathbf{T} \ , \tag{26.1}$$

where **T** is the transition matrix associated with actual interactions as opposed to just undisturbed propagation (corresponding to the identity in (26.1). Imposing the unitarity of the **S** matrix, i.e.

$$\mathbf{S}^{\dagger}\mathbf{S} = \mathbf{1} \ , \tag{26.2}$$

translates into a condition on **T** given by

$$-i\left(\mathbf{T} - \mathbf{T}^{\dagger}\right) = \mathbf{T}^{\dagger}\mathbf{T} \ . \tag{26.3}$$

Let us now consider the transition matrix element between two-particle states. For this, it is useful to remember that we are using a Lorentz invariant normalization for momentum states given by

$$|\mathbf{q_i}\rangle = \sqrt{2E_i}\, a^\dagger_{q_i}\, |0\rangle \ , \tag{26.4}$$

such that

$$\langle \mathbf{q_i}|\mathbf{q_j}\rangle = 2E_i\,(2\pi)^3\delta^{(3)}(\mathbf{q_i}-\mathbf{q_j}) \ . \tag{26.5}$$

Then, the one particle identity can be written as

$$\mathbf{1} = \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{2E_q}\,|\mathbf{q}\rangle\langle\mathbf{q}| \ . \tag{26.6}$$

Inserting (26.6) for each intermediate state $|\mathbf{q_i}\rangle$, we can write the matrix element of the right hand side of (26.3) between initial and a final two-particle states as

$$\langle \mathbf{p_1p_2}|\mathbf{T}^\dagger\mathbf{T}|\mathbf{k_1k_2}\rangle = \sum_n \left(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i}\right)\langle \mathbf{p_1p_2}|\mathbf{T}^\dagger|\{\mathbf{q_i}\}\rangle\langle\{\mathbf{q_i}\}|\mathbf{T}|\mathbf{k_1k_2}\rangle \ . \tag{26.7}$$

Here, the $\mathbf{k_i}$ and $\mathbf{p_i}$ (i=1,2) are the initial and final momenta respectively. We also inserted a complete set of intermediate states

$$|\{\mathbf{q_i}\}\rangle \equiv |\mathbf{q_1}\ldots\mathbf{q_n}\rangle \ . \tag{26.8}$$

Recalling that we define the amplitude by extracting the momentum conservation delta function from the transition matrix element

$$\langle\{\mathbf{q_i}\}|\mathbf{T}|\mathbf{k_1k_2}\rangle = (2\pi)^4\delta^{(4)}(k_1+k_2-\sum_i q_i)\,\mathcal{A}(k_1k_2\to\{q_i\}) \ , \tag{26.9}$$

where the amplitude $\mathcal{A}$ refers to the process with initial state $|k_1k_2\rangle$ going to the *final* state $|\{q_i\}\rangle$. Analogously, we can write

$$\langle \mathbf{p_1p_2}|\mathbf{T}^\dagger\{\mathbf{q_i}\}\rangle = (2\pi)^4\delta^{(4)}(p_1+p_2-\sum_i q_i)\,\mathcal{A}^\dagger(p_1p_2\to\{q_i\}) \ . \tag{26.10}$$

Then, the unitarity condition (26.3) on $\mathbf{T}$ can be expressed as

$$\begin{aligned} -i\left(\mathcal{A}(k_1k_2\to p_1p_2)-\mathcal{A}^\dagger(k_1k_2\to p_1p_2)\right) &= \sum_n\left(\prod_{i=1}^n\int\frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i}\right)\times(2\pi)^4\delta^{(4)}(k_1+k_2-\sum_i q_i) \\ &\times\ \mathcal{A}^\dagger(p_1p_2\to\{q_i\})\,\mathcal{A}(k_1k_2\to\{q_i\})\ . \end{aligned} \tag{26.11}$$

In the expression above we have canceled the additional delta functions since

$$\delta^{(4)}(p_1+p_2-\sum_i q_i)=\delta^{(4)}(p_1+p_2-k_1-k_2)\ , \tag{26.12}$$

once the delta function in (26.11) is enforced. We notice that in the right hand side of (26.11) the states $|\{q_i\}\rangle$ now describe the final states of the amplitudes from the initial states $|k_1k_2\rangle$ and $|p_1p_2\rangle$. The phase space for these states, which we just call f for short, is just

$$d\Pi_f\equiv\prod_{i=1}^n\frac{d^3q}{(2\pi)^3}\frac{1}{2E_i}\ . \tag{26.13}$$

With this definition we rewrite (26.11) as

$$\boxed{2\,\mathrm{Im}\left[\mathcal{A}(a\to b)\right]=\sum_f\int d\Pi_f\,\mathcal{A}^\dagger(b\to f)\,\mathcal{A}(a\to f)}\ , \tag{26.14}$$

where we defined $a=(k_1,k_2)$ as the label of the initial state and $b=(p_1,p_2)$ as the one corresponding to the final state. The expression above is a general form of the optical theorem. It states that the imaginary part of the amplitude for a process can be obtained as the product of the amplitudes for these states to reach all possible *real* intermediate states, integrated over their available phase space. It is schematically shown in Figure 26.1.

To arrive at the more common form of the optical theorem, we consider the *forward* amplitude, i.e. the amplitude for $a=b$ (or for the two-particle example, for $(k_1,k_2)=(p_1,p_2)$). For this case (26.14) takes the form

$$2\mathrm{Im}\left[\mathcal{A}(a\to a)\right]=\sum_f\int d\Pi_f\,|\mathcal{A}(a\to f)|^2\ . \tag{26.15}$$

Here we recall that the differential cross section for a process with initial state $a=(k_1,k_2)$ and final state $f=\{q_i\}$ is given by

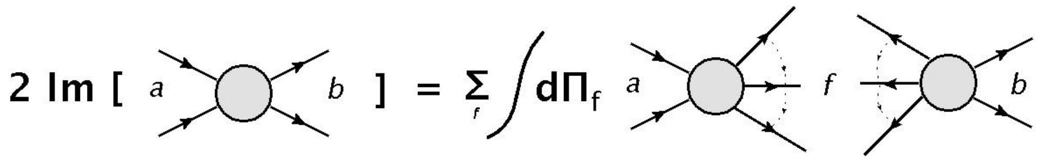


Figure 26.1: The optical theorem. The imaginary part of the amplitude between states a and b is given by the product of the amplitudes from a and b to all available intermediate states f, properly integrated over their phase space.

$$d\sigma_{a\to f} = \frac{(2\pi)^4\delta^{(4)}(k_1+k_2-\sum_i q_i)}{2E_1 2E_2\,|v_1-v_2|}\,d\Pi_f\,|\mathcal{A}(a\to f)|^2\ . \tag{26.16}$$

Then, in the center of momentum (CM) frame, $\mathbf{k_1} = -\mathbf{k_2}$ which results in

$$2E_1\,2E_2\,|v_1-v_2| = 4k_{\rm CM}\,E_{\rm CM}\ , \tag{26.17}$$

with $k_{\rm CM} = |\mathbf{k_1}| = |\mathbf{k_2}|$. Then (26.15) now reads

$$\boxed{\mathrm{Im}\,[\mathcal{A}(a\to a)] = 2k_{\rm CM}\,E_{\rm CM}\,\sigma_{\rm total}(a\to f)}\ , \tag{26.18}$$

which is the more known form of the optical theorem: the imaginary part of the forward scattering amplitude is given by the total cross section, i.e. the cross section o initial state a into all possible available final states f.

26.2 Unstable Particles

As we saw at the beginning of the renormalization program, the fully renormalized two point function is obtained by resumming the 1PI diagrams. This is depicted in Figure 26.2

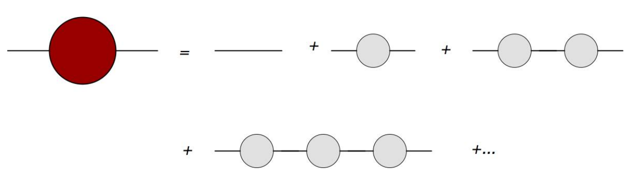
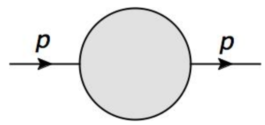


Figure 26.2: Resummation of the two-point function 1PI diagrams leading to the corrected propagator.

We recall that the 1PI diagram entering above is given by $-i\Sigma(p^2)$, where p is the external momentum flowing through the two point function, as shown below.



$$\equiv -i\Sigma(p^2) \ ,$$

The resummation results in

$$\Delta_F(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \ . \tag{26.19}$$

We had redefined the physical mass by $m^2 = m_0^2 + \Sigma(p_r^2)$ where p_r was the momentum where the renormalization condition was imposed. However, as we saw earlier, it is possible that the amplitude $\Sigma(p^2)$ develops an imaginary part. This is the case when the intermediate states in the diagrams contributing to $\Sigma(p^2)$ can be on the mass shell, i.e. the intermediate state can be real rather than virtual. This gives rise to a discontinuity that results in an imaginary part for $\Sigma(p^2)$, as prescribed by the optical theorem (26.14). Then the mass renormalization should be defined by

$$m^2 = m_0^2 + \mathrm{Re}\left[\Sigma(p_r^2)\right] \ , \tag{26.20}$$

whereas the renormalized propagator should now read

$$\Delta_F(p) = \frac{i}{p^2 - m^2 - i\mathrm{Im}\left[\Sigma(p^2)\right]} \ , \tag{26.21}$$

where now the propagator includes the imaginary part of the two point function. We can consider the two point function amplitude $-i\Sigma(p^2)$ as the amplitude for a forward process from a state a corresponding to a single particle of momentum p, i.e.

$$-i\Sigma(p^2) \equiv i\mathcal{A}(a \to a)(p^2) \ . \tag{26.22}$$

But the optical theorem says that

$$\mathrm{Im}\left[\Sigma(p^2)\right] = -\mathrm{Im}\left[\mathcal{A}(a \to a)(p^2)\right] = -\frac{1}{2}\sum_f \int d\Pi_f \, |\mathcal{A}(a \to f)|^2 \ . \tag{26.23}$$

Here the states labeled f are the ones accessible from the one-particle state a, i.e. the imaginary part of the two point function comes from the existence of states to which the propagating particle can decay.

If the particle in question is produced in the s channel, as shown in Figure 26.3

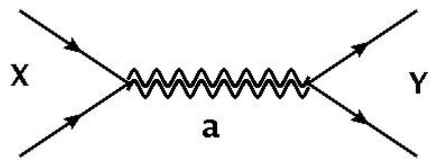


Figure 26.3: Feynman diagram of the production of an unstable particle in the s channel.

then, for $p^2 = s$ close to m^2, the production cross section will behave like

$$\sigma(X \to a \to Y) \sim \left| \frac{1}{s - m^2 - i\mathrm{Im}\left[\Sigma(s)\right]} \right|^2 , \tag{26.24}$$

It is tempting to identify the expression above with the Breit-Wigner shape of a resonance. In general this is not possible since $s \neq m^2$. However, if the imaginary part is small this results in a very small interval for s around m^2 (a narrow resonance). Then, we can approximate $p^2 \simeq m^2$ in the expression (26.23). With this approximation in mind, and recalling the form of the decay width of a particle of mass m to all available states f

$$\Gamma_a = \frac{1}{2m} \sum_f \int d\Pi_f \left|\mathcal{A}(a \to f)\right|^2 , \tag{26.25}$$

we arrive at

$$\mathrm{Im}\left[\Sigma(m^2)\right] = -m\,\Gamma_a \ . \tag{26.26}$$

Thus, the form of the propagator of a narrow unstable particle is

$$\Delta_F(p) = \frac{i}{p^2 - m^2 + im\Gamma_a} , \tag{26.27}$$

where Γ_a is the total width of the decaying particle. Now the cross section is

$$\sigma(X \to a \to Y) \sim \left| \frac{1}{s - m^2 + im\Gamma_a} \right|^2 = \frac{1}{(s-m^2)^2 + m^2\Gamma_a^2} , \tag{26.28}$$

which has the characteristic Breit-Wigner shape with width Γ_a.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 7.3.