Lecture 25

Calculation of the QED Counterterms

In order to complete the renormalization procedure for QED, we will now compute the counterterms δ_1, δ_2 and δ_3. But as we saw at the end of the previous lecture, we only need to compute δ_3 and *either* δ_1 or δ_2 on account of the relationship $\delta_1 = \delta_2$ resulting from gauge invariance/current conservation. We will then compute δ_2 and δ_3 and leave the computation of δ_1 as an excercise.

25.1 Electron Self-energy to One Loop Accuracy

We start with the computation of the counterterm δ_2 to one loop order in QED. The relevant diagram is shown in Figure 25.1.

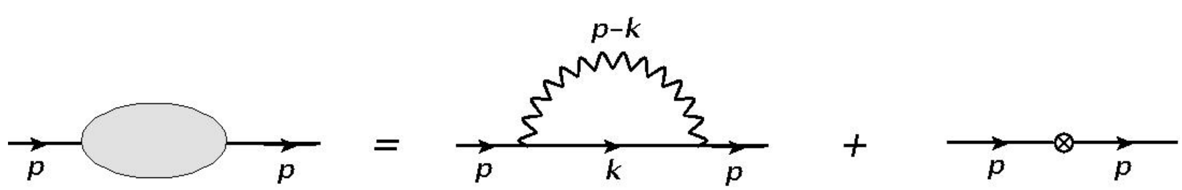

Figure 25.1: One loop contribution to the electron field and mass renormalization. The first diagram is the one loop contribution from the photon. The second is the couterterm contribution.

The electron self-energy then receives two contributions: the one loop contribution and the couterterms. This is

$$-i\Sigma(\not{p}) = -i\Sigma(\not{p})_{\text{loop}} + i(\delta_2 \not{p} - \delta m) \ , \tag{25.1}$$

where in the last term we used the Feynman rules for the fermion line couterterm we derived in the previous lecture.

We now want to compute the loop contribution $-i\Sigma(\not{p})_{\rm loop}$, which is given by

$$-i\Sigma(\not{p})_{\rm loop} = (-ie)^2 \int \frac{d^4k}{(2\pi)^2}\,\gamma^\mu\,\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\,\gamma^\nu\,\frac{-ig_{\mu\nu}}{(p-k)^2+i\epsilon}\ , \tag{25.2}$$

As usual, we start with the Feynman parametrization. The product of the two denominators can be written as

$$\begin{aligned}\frac{1}{k^2-m^2+i\epsilon}\,\frac{1}{(p-k)^2+i\epsilon} &= \int_0^1 dx\,\frac{1}{\left[x(p-k)^2+(1-x)(k^2-m^2\right]^2} \\ &= \int_0^1 dx\,\frac{1}{\left[k^2-2xp+xp^2-(1-x)m^2\right]^2}\ .\end{aligned} \tag{25.3}$$

Defining

$$D \equiv (k-xp)^2 - x^2p^2 + xp^2 - (1-x)m^2\ , \tag{25.4}$$

we can rewrite

$$D = \ell^2 - a^2\ , \tag{25.5}$$

where we defined

$$\ell \equiv k - xp\ , \tag{25.6}$$

and

$$a^2 \equiv (1-x)m^2 - x(1-x)p^2\ . \tag{25.7}$$

Then, we arrive at

$$-i\Sigma(\not{p})_{\rm loop} = (-ie)^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\,\gamma^\mu\,\frac{(\not{\ell}+x\not{p}+m)}{\left[\ell^2-a^2+i\epsilon\right]^2}\gamma_\mu\ , \tag{25.8}$$

In order to perform the integral we should be a bit careful. Since we will use dimensional regularization, we will recall some gamma matrix identities. First, we recall the identity

$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p} \ . \tag{25.9}$$

Next, we notice that the linear term in ℓ in (25.8) does not contribute since it is odd and we integrate ℓ over all values. Finally, in d dimensions we have

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= d \\
\gamma^\mu \not{p} \gamma_\mu &= -(d-2)\not{p} \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \ .
\end{aligned} \tag{25.10}$$

Using $d = 4 - \epsilon$, we see that the equalities above introduce an ϵ dependence in the numerator in (25.8). Then we must be careful when multiplying by $2/\epsilon$. We then obtain

$$-i\Sigma(\not{p})_{\rm loop} = -e^2 \mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{-(2-\epsilon)x\not{p} + (4-\epsilon)m}{\left[\ell^2 - a^2 + i\epsilon\right]^2} \ , \tag{25.11}$$

where we introduced the arbitrary renormalization scale μ with the appropriate power of ϵ to restore the original units. If we separate the d dimensional integral in (25.11) as

$$\begin{aligned}
I_d &\equiv \mu^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[\ell^2 - a^2 + i\epsilon\right]^2} = i\,\mu^\epsilon \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{\left[\ell_E^2 + a^2 - i\epsilon\right]^2} \\
&= i\frac{\pi^{d/2}}{(2\pi)^d}\,\mu^\epsilon\,\Gamma(\epsilon/2)\,\frac{1}{(a^2)^{\epsilon/2}} \ .
\end{aligned} \tag{25.12}$$

Inserting this in (25.11) we obtain

$$-i\Sigma(\not{p})_{\rm loop} = -ie^2 \frac{\mu^\epsilon}{(4\pi)^{d/2}} \int_0^1 dx\,\Gamma(\epsilon/2)\,\frac{(4-\epsilon)m - (2-\epsilon)x\not{p}}{\left[(1-x)m^2 - x(1-x)p^2\right]^{\epsilon/2}} \ . \tag{25.13}$$

With this result in hand we are now ready to make use of (25.1) to obtain the counterterms to this order in perturbation theory. To this end we must impose the renormalization condition on $\Sigma(\not{p})$. We choose $\not{p} = m$ as the renormalization point. Then we impose

$$\Sigma(m) = 0 \ , \tag{25.14}$$

resulting in

$$m\delta_2 - \delta m = \Sigma(m)_{\rm loop} \ . \tag{25.15}$$

The second renormalization condition fixes the residue and is

$$\left.\frac{d\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m} = 0 \ , \tag{25.16}$$

which results in

$$\delta_2 = \left.\frac{d\Sigma(\not{p})_{\rm loop}}{d\not{p}}\right|_{\not{p}=m} \ . \tag{25.17}$$

This results in

$$\begin{aligned}\delta_2 = \frac{e^2}{(4\pi)^{d/2}} \qquad & \int_0^1 dx\,\left(\frac{2}{\epsilon} - \gamma_E - \ln\left(\frac{(1-x)m^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\right) \\ & \times \left\{-(2-\epsilon)x + \left(\frac{\epsilon}{2}\right) x \,\frac{[4-2x-\epsilon(1-x)]}{(1-x)}\right\} \ .\end{aligned} \tag{25.18}$$

In the expression above we left terms of order ϵ since they result in finite contributions once they are multiplied by $2/\epsilon$. Replacing (25.18) in (25.15) we obtain

$$\begin{aligned}\delta m = m\,\frac{e^2}{16\pi^2} \qquad & \int_0^1 dx\,\left(\frac{2}{\epsilon} - \gamma_E - \ln\left(\frac{(1-x)m^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\right) \\ & \times \frac{(\epsilon-2)x((1+\epsilon)-2)\,[4-2x-\epsilon(1-x)]}{2(1-x)} \ .\end{aligned} \tag{25.19}$$

An important feature of the result in (25.19) is that

$$\delta m \propto m \ , \tag{25.20}$$

where m is the mass that appears in the lagrangian. This is true up to a $\log m$ that appears in (25.19) coming from the first factor inside the integral. This can be the bare mass m_0 or the renormalized mass m given that their difference interoduces a term in δm that is of higher order in e. The main point however is that since the correction to

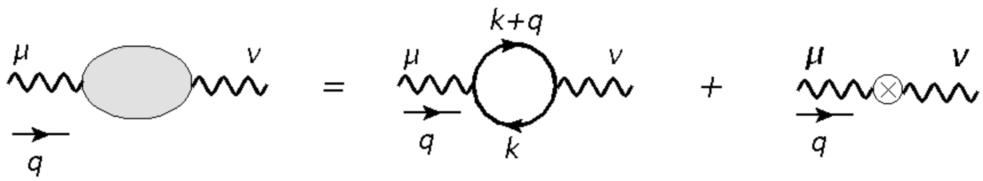

Figure 25.2: One loop contribution to the photon field renormalization. The first diagram is the one loop contribution from the fermions. The second is the couterterm contribution.

the fermion mass vanishes with the fermion mass in the theory, if this is zero in $\mathcal{L}$ it will remain zero to all orders in perturbation theory. This is another reflexion of chiral symmetry: if $m = 0$ in $\mathcal{L}$ it will not be induced via loop corrections and therefore chiral symmetry is respected to all orders even in the presence of interactions.

25.2 Photon Field Renormalization

To obtain the photon self-energy we take the same steps as with the fermion case above. The relevant diagrams are shown in Figure 25.2.

We need to compute the loop diagram. This is given by

$$\begin{aligned} i\Pi^{\mu\nu}_{\text{loop}}(q^2) &= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^2} \,\mathrm{Tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not{k} - \mathrm{m}} \gamma^\nu \frac{\mathrm{i}}{\not{k} + \not{q} - \mathrm{m}}\right] \\ &= -e^2 \int \frac{d^4}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \,\frac{\not{k} + \mathrm{m}}{\mathrm{k}^2 - \mathrm{m}^2}\, \gamma^\nu \,\frac{\not{k} + \not{q} + \mathrm{m}}{(\mathrm{k} + \mathrm{q})^2 - \mathrm{m}^2}\right] \;, \end{aligned} \tag{25.21}$$

where in the first line the overall sign comes from the presence of a closed fermion loop. We now introduce a Feynman parametrization and using the properties of the gamma matrices we obtain

$$i\Pi^{\mu\nu}_{\text{loop}}(q^2) = -4e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{N^{\mu\nu}}{\left[x((k+q)^2 - m^2) + (1-x)(k^2 - m^2)\right]^2} \;, \tag{25.22}$$

where we defined the tensor

$$N^{\mu\nu} = k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right] \;. \tag{25.23}$$

The denominator can be written as

$$\begin{aligned} D &\equiv k^2 + 2xk \cdot q + xq^2 - m^2 \\ &= (k + xq)^2 - x^2q^2 + xq^2 - m^2 \\ &\equiv \ell^2 - a^2 \ , \end{aligned} \tag{25.24}$$
$$\tag{25.25}$$

where in the last line we defined

$$\ell \equiv k + xq \ , \tag{25.26}$$

and

$$a^2 \equiv m^2 - x(1-x)q^2 \ . \tag{25.27}$$

Then, (25.22) can be expressed as

$$i\Pi^{\mu\nu}_{\text{loop}}(q^2) = -4e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{[\ell^2 - a^2]^2} \ . \tag{25.28}$$

The numerator tensor in terms of the new integration variable ℓ is

$$\begin{aligned} N^{\mu\nu} &= 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left(m^2 + x(1-x)q^2\right) \\ &+ \text{terms linear in } \ell \end{aligned} \tag{25.29}$$

In order to proceed we will use dimensional regularization. For this purpose we should remember that in going to d dimensions we have

$$g^{\mu\nu}g_{\mu\nu} = d \ . \tag{25.30}$$

Also, when integrating the terms that are linear in ℓ in $N^{\mu\nu}$ vanish and we can make the replacement

$$\ell^\mu\ell^\nu \to \frac{1}{d}g^{\mu\nu}\ell^2 \ . \tag{25.31}$$

Proceeding with the usual steps in dimensional regularization then results in

$$i\Pi^{\mu\nu}_{\rm loop}(q^2) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \, i\Pi_{\rm loop}(q^2) \ , \tag{25.32}$$

where we have defined

$$\Pi_{\rm loop}(q^2) = -\frac{8e^2}{(4\pi)^{d/2}} \, \mu^\epsilon \int_0^1 x(1-x)dx \, \frac{\Gamma(\epsilon/2)}{(a^2)^{\epsilon/2}} \tag{25.33}$$

If we add both contributions in Figure 25.2 we have

$$i\Pi^{\mu\nu}(q^2) = i\Pi^{\mu\nu}_{\rm loop}(q^2) - i\left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \delta_3 \ . \tag{25.34}$$

Imposing on (25.34) the renormalization condition

$$\Pi^{\mu\nu}(q^2 = 0) = 0 \ , \tag{25.35}$$

we arrive at

$$\boxed{\delta_3 = \Pi_{\rm loop}(0) = -\frac{8e^2}{16\pi^2} \int_0^1 dx x(1-x) \left\{ \frac{2}{\epsilon} - \gamma_E - \ln \frac{m^2}{\mu^2} + \ldots \right\}} \ , \tag{25.36}$$

This concludes the computation of the QED couterterms to one loop accuracy. The couterterm associated to the coupling, δ_1 is already obtained by using the Ward identity resulting in $\delta_1 = \delta_2$.

25.3 Charge Renormalization

We had defined the renormalized charged e by the relation

$$e \, Z_1 = Z_3^{1/2} \, Z_2 \, e_0 \ , \tag{25.37}$$

where e_0 is the bare (unrenormalized) charge. Here Z_2 and Z_3 defined the fermion and photon field renormalizations (as well as the associated couterterms $\delta_2 = 1 - Z_2$ and $\delta_3 = 1 - Z_3$). Z_1 defined the coupling renormalization and in particular the associated counterterm, through

$$Z_1 = 1 + \delta_1 \ . \tag{25.38}$$

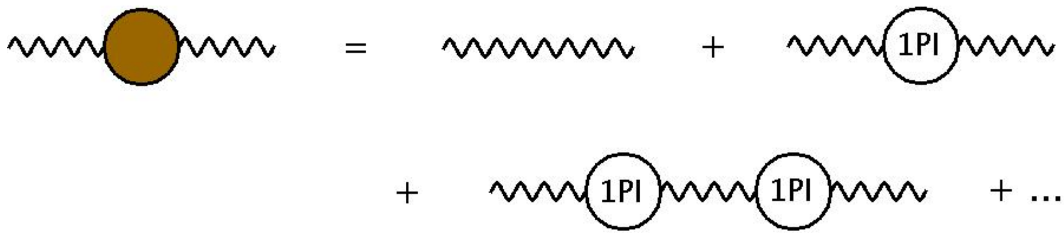

Figure 25.3: One loop resummation of the 1PI diagrams correcting the photon field.

However, the Ward identity derived in the previous lecture, i.e.

$$Z_1 = Z_2 \ , \tag{25.39}$$

guarantees that the renormalization of the charge e only depends on the renormalization of the photon field:

$$\boxed{e = Z_3^{1/2}\, e_0} \ . \tag{25.40}$$

So we arrive at a result that states that the renormalization of the electric charge is universal since it depends only on the photon field renormalization, and not on details of the external fermions that could appear in Z_1 and Z_2 such as their masses. Just as we would expect, the universality of the renormalized QED coupling is the result of gauge invariance via the Ward identity (25.39).

We will now explore another consequence of the renormalization of the coupling: its momentum dependence. This is similar to what we observed in scalar theories before. As a first step we must re-sum the correction to the photon propagator in order to obtain its effect in the coupling. This is schematically shown in Figure 25.3.

We interpret the left-hand side as the corrected photon propagator. Then we write

$$-i\frac{g_{\mu\nu}}{q^2}\, F(q^2) = -i\frac{g_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i\left(g^{\rho\sigma}q^2 - q^\rho q^\sigma\right)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \ldots \tag{25.41}$$

where on the left-hand side we define the form-factor $F(q^2)$ to contain all the corrections, and we used

$$\Pi(q^2) = \Pi_{\rm loop}(q^2) - \delta_3 \ , \tag{25.42}$$

which is the correction including the counterterm δ_3, and where $\Pi_{\rm loop}(q^2)$ is the one defined in (25.32) and given in (25.33). The fact that we are including the counterterm in the resummation signals that we are using the renormalized theory plus couterterms that we defined in the previous lecture. The expression in (25.41) goes up to the first 1PI loop diagram in Figure 25.3. To accommodate the higher order terms is convenient to define

$$\Delta^\rho_\nu \equiv \delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2} \ . \tag{25.43}$$

Then we have

$$-i\frac{g_{\mu\nu}}{q^2}\,F(q^2) = -i\frac{g_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\,\Delta^\rho_\nu\,\Pi(q^2) + \frac{-ig_{\mu\rho}}{q^2}\,\Delta^\rho_\sigma\,\Delta^\sigma_\nu\,\Pi^2(q^2) + \dots \tag{25.44}$$

But noticing that

$$\begin{aligned} \Delta^\rho_\sigma\,\Delta^\sigma_\nu &= \left(\delta^\rho_\sigma - \frac{q^\rho q_\sigma}{q^2}\right)\left(\delta^\sigma_\nu - \frac{q^\sigma q_\nu}{q^2}\right) = \delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2} - \frac{q^\rho q_\nu}{q^2} + \frac{q^\rho q_\nu}{q^2} \\ &= \delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2} = \Delta^\rho_\nu \ , \end{aligned} \tag{25.45}$$

we can rewrite (25.44) as

$$-i\frac{g_{\mu\nu}}{q^2}\,F(q^2) = -i\frac{g_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\,\Delta^\rho_\nu\left(\Pi(q^2) + \Pi^2(q^2) + \dots\right) \tag{25.46}$$

The Π series can be written as

$$\Pi(q^2) + \Pi^2(q^2) + \dots = \frac{1}{1-\Pi(q^2)} - 1 \ , \tag{25.47}$$

which when replaced in (25.46) results in

$$-i\frac{g_{\mu\nu}}{q^2}\,F(q^2) = -i\frac{g_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\,\Delta^\rho_\nu\,\frac{1}{1-\Pi(q^2)} - \frac{-ig_{\mu_\rho}}{q^2}\,\Delta^\rho_\nu \ . \tag{25.48}$$

We can isolate the terms proportional to products of the momentum q. This gives

$$-i\frac{g_{\mu\nu}}{q^2}\,F(q^2) = \frac{-ig_{\mu\rho}}{q^2}\,\frac{1}{1-\Pi(q^2)} + i\frac{q_\mu q_\nu}{q^2}\left(\frac{1}{1-\Pi(q^2)} - 1\right) \ . \tag{25.49}$$

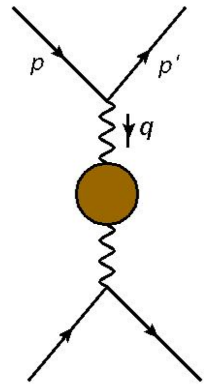

Figure 25.4: The corrected photon propagator in a fermion scattering process.

But the full photon propagator is always used to compute amplitudes involving conserved currents. These involve the product of two currents such as the two fermion currents in the diagram in Figure 25.4. When these conserved currents are contracted with the photon momentum q the result is zero. One way to see this is to consider what happens to the top vertex fermion current when contracted with the photon momentum:

$$\bar{u}(p')\gamma_\mu u(p)\, q^\mu = \bar{u}(p')\, (\not{p} - \not{p}')\, u(p) = 0 \ , \tag{25.50}$$

where in the last equality we use the Dirac equation for both $u(p)$ and $\bar{u}(p')$.

Then, only the first term in (25.49) contributes, resulting in

$$-i\frac{g_{\mu\nu}}{q^2}\, F(q^2) = \frac{-ig_{\mu\rho}}{q^2}\, \frac{1}{1-\Pi(q^2)} \ . \tag{25.51}$$

We see then that the one loop correction ot the photon propagator results in a q^2-dependence of an amplitude involving the product of two conserved currents. We will interpret this dependence as one acquired by the renormalized charge e, which is the coupling that appears in the vertices of this theory. The amplitude will always contain the combination

$$-i\frac{g_{\mu\nu}}{q^2}\, F(q^2)\, e^2 = \frac{-ig_{\mu\rho}}{q^2}\, e^2\, \frac{1}{1-\Pi(q^2)} \ , \tag{25.52}$$

which we can use to define a q^2-dependent renormalized charge as in

$$\boxed{e^2_{\text{eff.}}(q^2) \equiv \frac{e^2}{1-\Pi(q^2)}} \,, \tag{25.53}$$

which means that the effective charge is energy dependent due to the loop corrections. This is another example of an energy dependence of the parameters of the theory induced by the virtual quantum corrections. This is a measurable effect in QED and up to one loop accuracy is given by

$$\Pi(q^2) = \Pi_{\text{loop}}(q^2) - \delta_3 = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \,, \tag{25.54}$$

where we used $\alpha = e^2/4\pi$. First, we see that the quantity in (25.54) does not depend on the arbitrary renormalization scale μ, as it should since it is a physical quantity. Second, this logarithmic dependence will result in an increase of the value of e at higher energies. This is a general feature of abelian gauge theories. The energy dependence of couplings such as e as well as other parameters of the theory, are a hint of a deeper point: quantum field theories flow from one energy scale to another. We will study this aspect of quantum field theory more formally in the second part of the course when we talk about the renormalization group.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 10.3.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Section 7.1
- *Quantum Field Theory* , by M. Srednicki, Chapter 62.