Lecture 24

Renormalization of QED

We are now in a position to apply the renormalization procedure to QED. We will proceed by first defining the renormalized fields, coupling and mass so that we can write the theory in terms of renormalized parameters and the corresponding counterterms. The next step is to impose renormalization conditions so we can then compute the counterterms. However, since QED is a gauge theory there will be a subtlety that results in relations among counterterms. We will address this before we actually compute them. We start with the unrenormalized lagranian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} + \bar{\psi}_0\left(i\not{D} - m_0\right)\psi_0 \\ &= -\frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} + \bar{\psi}_0\left(i\not{\partial} - m_0\right)\psi_0 - e_0\bar{\psi}_0\gamma^\mu\psi_0 A^0_\mu\ . \end{aligned} \tag{24.1}$$

<u>Field Renormalization</u>:

We define the corrections to the fermion and gauge boson fields by

$$\psi_0 \equiv Z_2^{1/2}\,\psi \tag{24.2}$$

$$A^0_\mu \equiv Z_3^{1/2}\,A_\mu\ . \tag{24.3}$$

Here Z_2 and Z_3 result in corrections to the propagators of fermions and the photon respectively. Writing the propagators in terms of the renormalized fields defined in (24.2 and (24.3) results in

$$\frac{iZ_2}{\not{p} - m_0}\ , \qquad\qquad \frac{-iZ_3\,g_{\mu\nu}}{q^2}\ . \tag{24.4}$$

If we now rewrite the lagrangian in terms of renormalized fields we have

$$\mathcal{L} = -Z_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi} \left(i\partial\!\!\!/ - m_0 \right) \psi - e_0 Z_2 Z_3^{1/2} \bar{\psi} \gamma^\mu \psi A_\mu \ , \tag{24.5}$$

where we have just replaced ψ_0 and A_μ^0 by (24.2) and (24.3).
Coupling Renormalization:
We now introduce the renormalized coupling e through the relation

$$e\, Z_1 \equiv e_0\, Z_2\, Z_3^{1/2} \ . \tag{24.6}$$

The renormalization condition to fix e at a given value of the momentum is typically chosen so as to obtain a value coming from a precise experimental measurement. Atomic physics provides very good examples of this.

Mass Renormalization:
We finally define the renormalized mass m through the relation

$$m + \delta m \equiv Z_2\, m_0 \ . \tag{24.7}$$

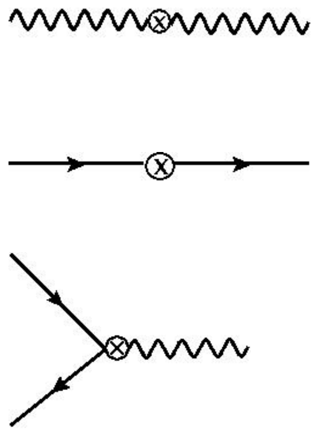
We are now in a position to define the counterterms through

$$\begin{aligned} Z_1 &\equiv 1 + \delta_1 \\ Z_2 &\equiv 1 + \delta_2 \\ Z_3 &\equiv 1 + \delta_3 \ , \end{aligned} \tag{24.8}$$

where the definitions suggest that we will consider the δ_i's as small perturbations coming from radiative loop corrections. Using (24.6), (24.7) and (24.8) in (24.5) we obtain the renormalized lagrangian with the appropriate counterterms.

$$\begin{aligned} \mathcal{L} \;=\;& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\partial\!\!\!/ - m \right) \psi - e \bar{\psi} \gamma^\mu \psi A_\mu \\ & -\frac{1}{4} \delta_3 F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\delta_2 \partial\!\!\!/ - \delta m \right) \psi - e\delta_1 \bar{\psi} \gamma^\mu \psi A_\mu \end{aligned} \tag{24.9}$$

The Feynman rules associated with the first line of (24.9) are the same as the ones derived earlier for the tree-level theory, only now we use the renormalized coupling e, the renormalized mass m and the renormalized electron and fóton fields. On the other hand,



$$= -i(g_{\mu\nu}q^2 - q_\mu q_\nu)\,\delta_3$$

$$= i(\delta_2\, \not{p} - \delta m)$$

$$= -ie\gamma_\mu\,\delta_1$$

Figure 24.1: Feynman rules for the counterterms in the renormalized QED lagrangian (24.9).

the second line of the equation above generates new Feynman rules associated with the counterterms. These are shown in Figure 24.

To understand the form of the photon two-point function counterterm it helps to notice that

$$-\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\delta_3\, A_\nu \left(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right) A_\mu \;, \tag{24.10}$$

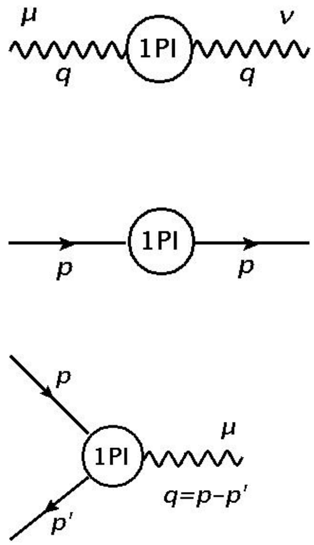
where we have integrated by parts. Going to momentum space this expression turns to the one at the top of Figure 24.

24.1 Renormalization Conditions

The next step is to impose renormalization conditions that will fix the computation of the couterterms δ_1, δ_2 δ_3 and δm to the desired order in perturbation theory. The one-particle irreducible (1PI) diagrams contributing to the propagators and the vertex define the appropriate momentum dependent structures. These are shown in the figure below and besides the 1PI diagrams to the desired order they also include the counterterms. Here $\Gamma^\mu(p,p')$ is some non-trivial Dirac structure containing γ matrices, momenta, etc.

We will choose the following renormalization conditions:

$$\boxed{\Sigma(\not{p} = m) = 0}\;, \tag{24.11}$$



$$= i(g_{\mu\nu}q^2 - q_\mu q_\nu)\,\Pi(q^2)$$

$$= -i\Sigma(\not{p})$$

$$= -ie\Gamma^\mu(p,p')$$

Figure 24.2: 1PI contributions to the photon and fermion propagators and the vertex.

fixes m, the renormalized mass.

$$\boxed{\left.\frac{d\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m} = 0}\,, \tag{24.12}$$

fixes the residue of the fermion propagator. Finally, the renormalization condition on the vertex

$$\boxed{-ieZ_1\Gamma^\mu(p-p'=0) = -ie\gamma^\mu}\,, \tag{24.13}$$

fixes the coupling, i.e. defines the electron charge e at $q^2 = 0$. As we will see later, we can compute the one-loop 1PI diagrams contributing to $\Pi(q^2)$, $\Sigma(\not{p})$ and $\Gamma^\mu(p,p')$. These plus the couterterm contributions from Figure 24 are forced to satisfy the renormalization conditions above. These determine the counterterms and define the renormalized parameters of the theory, i.e. m, e and the renormilazed fields ψ and A_μ. Then, armed with this knowledge we can compute any process to this order in perturbation theory and it will be finite and well defined, as we know from having performed the same procedure in the scalar case. However, in QED there is a new element: QED is a gauge theory, and gauge invariance imposes new restrictions on the renormalization procedure. These restrictions appear in the form of relations among some of the couterterms. We will consider these in what follows before we embark in the perturbative calculation of the counterterms.

24.2 The Ward Identities

In the presence of symmetries, and therefore of conserved currents, not all counterterms are independent. In what follows we will derive a generalized form of the Ward identity derived in Lecture 18, which will result in relations among counterterms for the case of a conserved vector current. In this more general version, we will *not require* the external momenta to be on shell [1].

We will be considering the case of a scalar field carrying a global $U(1)$ conserved charge, which will be simpler but will illustrate very well the more general case of QED. We start with the theory of a self-interacting complex scalar field with lagrangian

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \lambda\,(\phi^*\phi)^2 \ , \tag{24.14}$$

which is invariant under the global $U(1)$ transformations

$$\phi(x) \to e^{i\alpha}\,\phi(x)\ , \qquad\qquad \phi^*(x) \to e^{-i\alpha}\,\phi^*(x)\ , \tag{24.15}$$

where α is a real constant. The conserved vector current associated with (24.15) is

$$J_\mu = i\left[(\partial_\mu\phi^*)\phi - (\partial_\mu\phi)\phi^*\right] \ . \tag{24.16}$$

This is analogous to the QED charged fermion current, given by $e\bar{\psi}\gamma_\mu\psi$. Just as in the QED case we can write the three-point function for the insertion of the current between two fields. This is shown in Figure 24.3 below.

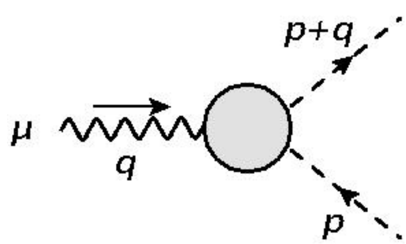


Figure 24.3: Insertion of the vector current in the theory of a complex scalar. This three-point function might then be coupled to an external vector source, such as the photon in QED.

The vector three-point function associated with the diagram in Figure 24.3 can then be coupled to an external vector field, such as the photon in QED. But for the purpose of

[1] This is the so called Ward-Takahashi identity. When considering on shell external momenta, we will recover the one obtained earlier.

deriving the Ward identities all we need is the form of the three-point function. It is given by

$$G_\mu(p,q) = \int d^4x\, d^4y\, e^{-iq\cdot x} e^{-ip\cdot y} \langle 0|T\, J_\mu(x)\phi(y)\phi^\dagger(0)|0\rangle \ , \tag{24.17}$$

where T denotes the time-ordered product. In (24.17) we took the third spacetime position to be the origin, i.e. $z = 0$. This is possible since the d^4z integral can be eliminated by momentum conservation, which we assume implicitly. The vector current carries momentum q_μ. If we multiply (24.17) by it we have

$$\begin{aligned} q^\mu\, G_\mu(p,q) &= \int d^4x\, d^4y\, i\partial_x^\mu \left(e^{-iq\cdot x - ip\cdot y}\right) \langle 0|T\, J_\mu(x)\phi(y)\phi^\dagger(0)|0\rangle \\ &= (-i)\int d^4x\, d^4y\, e^{-iq\cdot x - ip\cdot y}\, \partial_x^\mu \left(\langle 0|T\, J_\mu(x)\phi(y)\phi^\dagger(0)|0\rangle\right) \ . \end{aligned} \tag{24.18}$$

In the first line of (24.18) we wrote q^μ as $i\partial_x^\mu$ applied to the exponential. In the second line we integrated by parts so that the derivative with respect to the spacetime point x is applied on the three-point function in position space. But we should be careful how we apply this derivative since the current $J_\mu(x)$ is not the only x dependence there. There is a hidden x dependence in the time-ordering. To see this explicitly, we write

$$\begin{aligned} T\left(J_\mu(x)\phi(y)\phi^\dagger(0)\right) &= \theta(x_0 - y_0)\, J_\mu(x)\phi(y)\phi^\dagger(0) + \theta(y_0 - x_0)\, \phi(y)J_\mu(x)\phi^\dagger(0) \\ &+ \theta(x_0 - 0)\, J_\mu(x)\phi^\dagger(0)\phi(y) + \theta(0 - x_0)\, \phi^\dagger(0)J_\mu(x)\phi(y) \\ &+ \theta(y_0 - 0)\, \phi(y)\phi^\dagger(0)J_\mu(x) + \theta(0 - y_0)\, \phi^\dagger(0)\phi(y)J_\mu(x) \ . \end{aligned} \tag{24.19}$$

We now see that when applying ∂_x^μ to this expression, there are additional terms other than the divergence of the current $J_\mu(x)$. The result is

$$\begin{aligned} \partial_x^\mu T\left(J_\mu(x)\phi(y)\phi^\dagger(0)\right) &= T\left(\partial_x^\mu J_\mu(x)\phi(y)\phi^\dagger(0)\right) + T\left(\delta(x_0 - y_0)\left[J_0(x),\phi(y)\right]\phi^\dagger(0)\right) \\ &+ T\left(\delta(x_0)\left[J_0(x),\phi^\dagger(0)\right]\phi(y)\right) \ , \end{aligned} \tag{24.20}$$

where the δ functions appear as a result of taking the ∂_x^0 derivatives on the θ functions and the time-ordered operator is restored after taking the derivative in order to accommodate the time order of the third factor.

As a result we can write

$$\begin{aligned} q^\mu G_\mu(p,q) &= (-i)\int d^4x\, d^4y\, e^{-iq\cdot x - ip\cdot y}\,\langle 0|T\left(\delta(x_0-y_0)\,[J_0(x),\phi(y)]\,\phi^\dagger(0)\right)|0\rangle \\ &+ (-i)\int d^4x\, d^4y\, e^{-iq\cdot x - ip\cdot y}\,\langle 0|T\left(\delta(x_0)\,[J_0(x),\phi^\dagger(0)]\,\phi(y)\right)|0\rangle\ , \end{aligned} \tag{24.21}$$

where we have used current conservation. Next, we can use our knowledge of the commutators, from the non-trivial quantization condition

$$[\pi(\mathbf{x},t),\phi(\mathbf{x}',t)] = -i\delta^{(3)}(\mathbf{x}-\mathbf{x}')\ , \tag{24.22}$$

which for the theory in (24.14) corresponds to

$$[\partial_0\phi^\dagger(\mathbf{x},t),\phi(\mathbf{x}',t)] = -i\delta^{(3)}(\mathbf{x}-\mathbf{x}')\ . \tag{24.23}$$

Then, using the explicit form of the zero component of the vector current in (24.16), we obtain

$$[J_0(\mathbf{x},t),\phi(\mathbf{x}',t)] = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x},t)\ , \tag{24.24}$$

and

$$[J_0(\mathbf{x},t),\phi^\dagger(\mathbf{x}',t)] = -\delta^{(3)}(\mathbf{x}-\mathbf{x}')\phi^\dagger(\mathbf{x},t)\ . \tag{24.25}$$

Replacing these expressions above for the commutators into (24.21) we arrive at

$$\begin{aligned} q^\mu G_\mu(p,q) &= (-i)\int d^4x\, e^{-i(q+p)\cdot x}\,\langle 0|T\left(\phi(x)\phi^\dagger(0)\right)|0\rangle \\ &+ (+i)\int d^4y\, e^{-ip.\cdot y}\,\langle 0|T\left(\phi^\dagger(0)\phi(y)\right)|0\rangle\ . \end{aligned} \tag{24.26}$$

Noticing that on the right-hand side of (24.26) we have the propagators in momentum space, we can rewrite this expression as

$$-iq^\mu G_\mu(p,q) = \tilde{D}(p) - \tilde{D}(p+q) \ , \tag{24.27}$$

where the momentum space propagator is of course given by

$$\tilde{D}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \ . \tag{24.28}$$

The expression (24.27) is already a version of the Ward identities: it tells us that, when making use of current conservation and the explicit form of the current, the three-point function (or at least its contraction with external momentum) is related to the two-point function, i.e. the propagator. However, to state the Ward identity in terms of the vertex function, i.e. the one entering physical amplitudes, we need to amputate $G_\mu(p,q)$ by removing the external scalar propagators. We define the vertex function $\Gamma_\mu(p,q)$ by this amputation procedure

$$-iq^\mu\Gamma_\mu(p,q) \equiv \tilde{D}^{-1}(p+q)\left(-iq^\mu G_\mu(p,q)\right)\tilde{D}^{-1}(p) \ . \tag{24.29}$$

Inserting (24.29) into (24.27) we obtain the Ward identity in terms of the vertex function $\Gamma_\mu(p,q)$, given by

$$\boxed{-iq^\mu\Gamma_\mu(p,q) = \tilde{D}^{-1}(p+q) - \tilde{D}^{-1}(p)} \ . \tag{24.30}$$

This identity states that the vertex function and the propagator are related through the mere fact of vector current conservation. The Ward identity in (24.30) is then equivalent to current conservation.

We can derive a similar Ward identity for QED, relating the three point function to the fermion propagators. Of course, in this case, current conservation goes together with gauge invariance. As we will see below for the case of QED, this relationship between three- and two-point functions also results in relationships beween seemingly independent counterterms.

24.3 Ward Identity in QED and Counterterm Relations

The derivation of the Ward identity in QED proceeds very similarly to what we did in the previous section. The relevant diagram is the one in Figure 24.4, where we just replaced

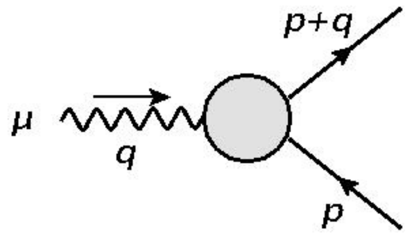


Figure 24.4: The insertion of the vector current in QED.

the scalar fields by fermions. The current is now

$$J_\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x) \ . \tag{24.31}$$

Just making use of (24.30) we obtain

$$-iq^\mu\Gamma_\mu(p,q) = S^{-1}(p+q) - S^{-1}(p) \ , \tag{24.32}$$

with $S(p)$ the fermion propagator in momentum space.
We can use the Ward identity (24.32) to obtain a relationship between the counterterms δ_1 and δ_2 associated with the vertex and fermion propagator. To do this we first recall the renormalization condition for the vertex function (24.13)

$$e\, Z_1\, \Gamma_\mu(q^2 \to 0) = e\gamma_\mu \ . \tag{24.33}$$

On the other hand, the renormalization condition (24.12) fixes the residue of the fermion propagator such that

$$S(p) = \frac{iZ_2}{\not{p} - m + i\epsilon} \ . \tag{24.34}$$

Imposing (24.32) results in

$$-iq^\mu Z_1^{-1}\gamma_\mu = (-i)Z_2^{-1}(\not{p}+\not{q} - m) - (-i)Z_2^{-1}(\not{p} - m) \ , \tag{24.35}$$

or

$$-i\not{q}Z_1^{-1} = -i\not{q}Z_2^{-1} \ , \tag{24.36}$$

which results in

$$\boxed{Z_1 = Z_2} \,, \tag{24.37}$$

The expression above is a direct consequence of the Ward identity and therefore of gauge invariance. It states that the renormalization of the charge and the residue of the fermion propagator are one and the same. This can be restated in terms of counterterms as

$$\boxed{\delta_1 = \delta_2} \,. \tag{24.38}$$

Then, computing one of these counterterms is enough to know the other one in QED. We will make use of this simplification in completing the renormalization procedure for QED.

Another important consequence of (24.37) can be seen by looking at (24.6), the definition of the renormalized charge e

$$eZ_1 = e_0 Z_2 Z_3^{1/2} \,. \tag{24.39}$$

We see that as a consequence of (24.37) this now reads

$$e = e_0 Z_3^{1/2} \,. \tag{24.40}$$

The expression above states that charge renormalization is entirely determined by the renormalization of the photon field and is independent of the fermion field and vertex renormalizations. Once again, this is a consequence of the Ward identity and therefore also of gauge invariance. As we will see later, when we compute the vertex and fermion field counterterms, these might depend on details such as the fermion masses. But since all fermion fields with the same charge must transform equally under gauge transformations, it is to be expected that the renormalized charge e should not depend on these details. The Ward identity makes sure this is the case: that dependence drops out when Z_1 and Z_2 cancel in (24.39).

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 7.4.

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Sections 7.1.3 and 8.4.1