# Lecture 23

# Regularization II: Dimensional Regularization

We will introduce another method of regularization that makes use of the fact that the problem of divergences can be ameliorated not only by increasing the power of momentum of the denominator, but also by decreasing the power in the numerator. In fact, if we make the number of dimensions $d$ small enough so that the $d^d k$ factor carries less powers of momentum, we can make any integral convergent even if they diverge for $d \to 4$. To implement this thinking we have to consider Feynman diagrams as analytic functions in the number of dimensions.

Let us start once again with the four-point function in $\phi^4$ theory as introduced in Lecture 20. Just as we did in Lecture 21, we focus on the s-channel diagram of Figure 20.5(a), and Figure 21.1, depicted again below in Figure 23.1 for reference.

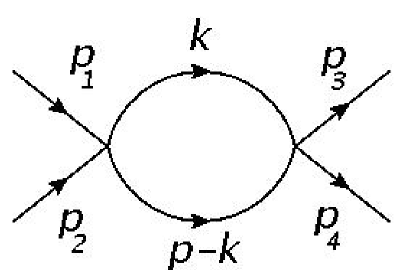


Figure 23.1: The s-channel one-loop correction to the four-point function $\Gamma(s, t, u)$ in $\lambda\phi^4$ theory. Here $p = p_1 + p_2 = p_3 + p_4$.

The integral of interest is

$$I_4 = \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 - m^2 + i\epsilon}\, \frac{1}{(p-k)^2 - m^2 + i\epsilon}\ , \tag{23.1}$$

where the external momentum is $p = p_1 + p_2 = p_3 + p_4$. As we discussed previously, this integral is logarithmically divergent. But if we analytically continue to a space of $d-1$ spatial dimensions and one time dimension, we now can write the momentum in the integral as

$$k^\mu = (k_0, k_1, k_2, \ldots, k_{d-1})\ , \tag{23.2}$$

for any $d$. On the other hand, the external momentum need not be changed, since it is not the source of divergences. So in the notation of $d$-dimensional Minkowski space it is given by

$$p^\mu = (p_0, p_1, p_2, p_3, 0, \ldots, 0)\ . \tag{23.3}$$

With this extension we now have a $d$-dimensional integral as

$$I_d = \mu^{4-d} \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2 - m^2 + i\epsilon}\, \frac{1}{(p-k)^2 - m^2 + i\epsilon}\ , \tag{23.4}$$

where we introduced the arbitrary mass scale $\mu$ in order to keep the correct units for the integral. The crucial observation is that the integral in (23.4) is *convergent* for $d < 4$. This is the only thing we need to assume about $d$ in order to carry out the calculation. It is the regulator of the method, which at the end of the calculation must be taking to its original value, i.e. at the end we take the $d \to 4$ limit to exhibit the divergences, in analogy to the $\Lambda \to \infty$ in the Pauli-Villars method studied earlier.

To continue the calculation we now go through the same steps we performed in the previous lecture: Feynman parametrization and Wick rotation. The relevant Feynman parametrization of (23.4) is

$$I_d = \mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx\ \frac{1}{\left[x\left((p-k)^2 - m^2\right) + (1-x)(k^2 - m^2)\right]^2}\ , \tag{23.5}$$

where we defined

$$\epsilon \equiv 4 - d\ . \tag{23.6}$$

The argument in brackets in the denominator can be written as

$$\begin{aligned} D &= k^2 - 2xk \cdot p + xp^2 - m^2 \\ &= (k - xp)^2 - x^2p^2 + xp^2 - m^2 \\ &= (k - xp)^2 + x(1 - x)p^2 - m^2 \ . \end{aligned} \tag{23.7}$$

Then, we can shift the integration variable defining

$$\ell^\mu \equiv k^\mu - x\, p^\mu \ , \tag{23.8}$$

such that the argument of the denominator can now be written simply as

$$D = \ell^2 - a^2 \ , \tag{23.9}$$

with

$$a^2 \equiv m^2 - x(1 - x)\, p^2 \ . \tag{23.10}$$

Putting it all together we rewrite the $d$-dimensional integral as

$$I_d = \mu^\epsilon \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[\ell^2 - a^2 + i\epsilon\right]^2} \ . \tag{23.11}$$

Performing the Wick rotation leaves the integral ready to be computed.

$$I_d = i\, \mu^\epsilon \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{\left[\ell_E^2 + a^2 - i\epsilon\right]^2} \ . \tag{23.12}$$

The part that is a little trickier now is to carry out the angular integration for any arbitrary value of $d$, even if it is not an integer, since we have

$$\int d^d\ell_E = \int d\Omega_d \quad \int d\ell_E\, \ell_E^{d-1} \ . \tag{23.13}$$

To do this we turn to a trick. We start by noticing the equality for a Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \ . \tag{23.14}$$

Then we can write

$$\left(\sqrt{\pi}\right)^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \int d^d x\, e^{-\sum_{i=1}^d x_i^2} \;, \tag{23.15}$$

where in the second equality we are assuming an euclidean $d$ dimensional space with $x^2 = x_1^2 + \cdots + x_d^2$, just as we have in (23.13). So now we can continue (23.15) to get

$$\begin{aligned} \left(\sqrt{\pi}\right)^d &= \int d\Omega_d \int dx\, x^{d-1}\, e^{-x^2} \\ &= \int d\Omega_d \int_0^\infty \frac{dx^2}{2}\, (x^2)^{d/2-1}\, e^{-x^2} \\ &= \int d\Omega_d\, \frac{1}{2}\, \Gamma(d/2) \;. \end{aligned} \tag{23.16}$$

In the last equality we used the integral definition of the Gamma function. From (23.16) we extract the value of the angular integral in a $d$-dimensional euclidean space, for arbitrary values of $d$

$$\boxed{\int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}} \;. \tag{23.17}$$

The integral in (23.12) is now given by

$$\begin{aligned} I_d &= i\mu^\epsilon \int_0^1 dx\, \frac{2\pi^{d/2}}{\Gamma(d/2)}\, \frac{1}{(2\pi)^d} \int_0^\infty d\ell_E\, \frac{\ell_E^{d-1}}{\left[\ell_E^2 + a^2 - i\epsilon\right]^2} \\ &= \frac{i\pi^{d/2}\,\mu^\epsilon}{(2\pi)^d\,\Gamma(d/2)} \int_0^1 dx \int_0^\infty d\ell_E^2\, \frac{\left(\ell_E^2\right)^{d/2-1}}{\left[\ell_E^2 + a^2 - i\epsilon\right]^2} \;. \end{aligned} \tag{23.18}$$

Using the properties of beta functions we have

$$\int_0^\infty dt\, \frac{t^{m-1}}{(t^2 + a^2)^n} = \frac{1}{(a^2)^{n-m}}\, \frac{\Gamma(m)\,\Gamma(n-m)}{\Gamma(n)} \;, \tag{23.19}$$

we arrive at

$$
\begin{aligned}
I_d &= \mu^\epsilon \frac{i\pi^{d/2}}{(2\pi)^d\,\Gamma(d/2)} \int_0^1 dx \frac{1}{(a^2 - i\epsilon)^{2-d/2}} \frac{\Gamma(d/2)\,\Gamma(2-d/2)}{\Gamma(2)} \\
&= \mu^\epsilon \frac{i\pi^{d/2}}{(2\pi)^d}\,\Gamma(2-d/2) \int_0^1 dx \frac{1}{(a^2)^{2-d/2}} \; . \qquad (23.20)
\end{aligned}
$$

In terms of $\epsilon$ the result for the $d$-dimensional integral is now

$$
\boxed{I_d = \mu^\epsilon \frac{i\pi^{d/2}}{(2\pi)^d}\,\Gamma(\epsilon/2) \int_0^1 dx \frac{1}{(a^2)^{\epsilon/2}}} \; . \qquad (23.21)
$$

But we know this integral is divergent. To see this, we notice that the Gamma function $\Gamma(z)$ has isolated poles for $z = 0, -1, -2, \ldots$. This means that $I_d$ has poles for $d = 4, 6, 8, \ldots$. In particular, we are interested in the $d \to 4$ limit, i.e. $\epsilon \to 0_+$. Around this limit we have the expansion

$$
\Gamma(\epsilon/2) \simeq \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \; , \qquad (23.22)
$$

where $\gamma_E = 0.5772\ldots$ is the Eüler-Mascheroni constant. On the other hand, the other relevant term has the expansion

$$
\frac{1}{(a^2)^{\epsilon/2}} \simeq 1 - \frac{\epsilon}{2}\ln a^2 + \mathcal{O}(\epsilon^2) \; . \qquad (23.23)
$$

Finally, we should use

$$
\mu^\epsilon = (\mu^2)^{\epsilon/2} \simeq 1 + \frac{2}{\epsilon}\ln\mu^2 + \ldots \; . \qquad (23.24)
$$

We should be careful when multiplying these factors since the logarithmic term in (23.23), although suppressed by $\epsilon/2$, will be lifted by the factor of $2/\epsilon$ in (23.22). Then

$$
\mu^\epsilon\,\Gamma(\epsilon/2)\,\frac{1}{(a^2)^{\epsilon/2}} \simeq \frac{2}{\epsilon} - \gamma_E - \ln\frac{a^2}{\mu^2} + \mathcal{O}(\epsilon) \; . \qquad (23.25)
$$

This gives us

$$
I_d = \frac{i}{16\pi^2} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma_E - \ln\frac{(m^2 - x(1-x)p^2)}{\mu^2} + \mathcal{O}(\epsilon) \right) \; , \qquad (23.26)
$$

which diverges as $\epsilon \to 0$, or as $d \to 4$. The one-loop contribution to the s-channel four-point function is

$$\begin{aligned}\Gamma(p^2) &= \frac{(-i\lambda)^2}{2}\,(-1)\,I_d \\ &= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma_E - \ln\frac{(m^2 - x(1-x)p^2)}{\mu^2} + \mathcal{O}(\epsilon)\right) \end{aligned} \tag{23.27}$$

If we now isolate the divergence by writing

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2)\ , \tag{23.28}$$

just as we did in the previous lecture, we obtain

$$\boxed{\Gamma(0) = \frac{i\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E - \ln\frac{m^2}{\mu^2}\right)}\ . \tag{23.29}$$

By comparing this result with what we obtained using the Pauli-Villars method

$$\Gamma(0) \simeq \frac{i\lambda^2}{32\pi^2}\,\ln\frac{\Lambda^2}{m^2}\ , \tag{23.30}$$

we can see that the pole in $\epsilon \to 0$ in dimensional regularization can be identified with a logarithmic divergence with the cutoff $\Lambda$ by making the replacement

$$\left(\frac{2}{\epsilon} - \gamma_E\right) \leftrightarrow \ln\Lambda^2\ . \tag{23.31}$$

It is straightforward to see that the finite piece of the four-point function is given by

$$\tilde{\Gamma}(p^2) = \frac{i\lambda}{32\pi^2}\int_0^1 dx\,\ln\left(\frac{m^2}{m^2 - x(1-x)p^2}\right)\ , \tag{23.32}$$

which is independent of the arbitrary scale $\mu$ and should be in agreement with the one obtained by the Pauli-Villars method. The dependence of the counterterm $\delta\lambda$ on the scale $\mu$ will play an important role when we consider the momentum dependence of renormalized parameters of a theory. In fact, we will call $\mu$ the renormalization scale and we will see that the coefficient of the logarithm of this scale determines the speed of variation of couplings with momentum, the $\beta$ function.

Next we want to compute the counterterms of the two-point function illustrated in Figure 23.2. As we did in previous lectures, we write this as

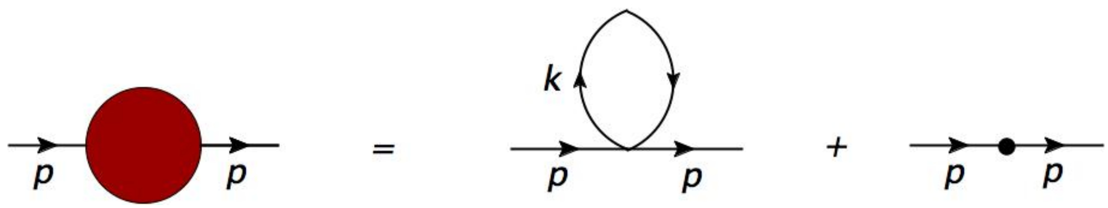


Figure 23.2: The 1PI diagrams contributing to the two-point function to $\mathcal{O}(\lambda)$. The last diagram corresponds to the counterterms.

$$-i\Sigma(p^2) = \frac{-i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} + i(\delta z p^2 - \delta m^2)\ . \tag{23.33}$$

This time we regularize the integral using dimensional regularization.

$$\begin{aligned} I_d^{(2)} &\equiv \frac{-i\lambda}{2}\,\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2} \\ &= \frac{-i\lambda}{2(2\pi)^d}\,\mu^\epsilon \int d\Omega_d \int_0^\infty dk_E\,\frac{k_E^{d-1}}{k_E^2+m^2-i\epsilon}\ , \end{aligned} \tag{23.34}$$

where in the last step we Wick rotated to euclidean momentum space, and we notice that due to the fact that there s only one propagator there is no need for a Feynman parametrization. Putting the integral in the form of a beta function we obtain

$$\begin{aligned} I_d^{(2)} &= \frac{-i\lambda}{2(2\pi)^d}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,\mu^\epsilon \int_0^\infty (1/2)dk_E^2\,\frac{(k_E^2)^{d/2-1}}{k_E^2+m^2-i\epsilon} \\ &= \mu^\epsilon\,\frac{-i\lambda\pi^{d/2}}{2(2\pi)^d\Gamma(d/2)}\,\frac{1}{(m^2)^{1-d/2}}\,\frac{\Gamma(d/2)\,\Gamma(1-d/2)}{\Gamma(1)} \\ &= \mu^\epsilon\,\frac{-i\lambda\pi^{d/2}}{2(2\pi)^d}\,\frac{1}{(m^2)^{1-d/2}}\,\Gamma(1-d/2)\ , \end{aligned} \tag{23.35}$$

where in the step before last we used the form of the beta function for $m = d/2$ and $n = 1$. The result clearly has poles for $d = 4$ but also $d = 2$. In order to make the expansion around the four-dimensional divergence more transparent we make use of one the properties of the gamma functions. They satisfy

$$z\,\Gamma(z) = \Gamma(z+1)\ . \tag{23.36}$$

Then in our case we can write

$$(1-d/2)\,\Gamma(1-d/2) = \Gamma(2-d/2) = \Gamma(\epsilon/2)\ , \tag{23.37}$$

so we can make the replacement

$$\Gamma(1-d/2) = \frac{\Gamma(\epsilon/2)}{1-d/2}\ . \tag{23.38}$$

Finally, we notice that

$$(m^2)^{1-d/2} = (m^2)^{-1}\,(m^2)^{\epsilon/2} \simeq (m^2)^{-1}\left(1+\frac{\epsilon}{2}\ln m^2+\dots\right)\ . \tag{23.39}$$

Then, expanding around $d\to 4$ limit we obtain

$$I_d^{(2)} = \frac{i\lambda}{32\pi^2}\,m^2\left(\frac{2}{\epsilon}-\gamma_E-\ln\frac{m^2}{\mu^2}+\mathcal{O}(\epsilon)\right) \tag{23.40}$$

Finally, imposing the renormalization condition, e.g. on $\Sigma(p^2=0)$, we obtain the mass counterterm given by

$$\boxed{\delta m^2 = \frac{\lambda}{32\pi^2}\,m^2\left(\frac{2}{\epsilon}-\gamma_E-\ln\frac{m^2}{\mu^2}+\mathcal{O}(\epsilon)\right)}\ . \tag{23.41}$$

Once again, just as for $\delta\lambda$, we notice that the counterterm $\delta m^2$ depends on the scale $\mu$ and that the coefficient of this logarithm is the same as that for the divergence. As mentioned above, this will play an important role in determining the renormalization flow of the parameters of the theory.

If we compare it with the one obtained using Pauli-Villars

$$(\delta m^2)_{PV} \simeq \frac{\lambda}{32\pi^2}\,\Lambda^2\ , \tag{23.42}$$

with $\Lambda$ the cutoff to be taken to infinity, we see that the structure of the divergence in the two-point function in dimensional regularization is somewhat different from that of the four-point function, despite the fact that the divergence looks the same (i.e. it goes as $2/\epsilon$). In this case we seem to need to make the replacement

$$\left(\frac{2}{\epsilon} - \gamma_E\right) \leftrightarrow \frac{\Lambda^2}{m^2} , \tag{23.43}$$

in order to match the two answers. This just tells us that we should be careful when trying to extract information about what kind of divergence occurs in dimensional regularization. In this sense, we see that in order to better evaluate the degree of sensitivity of a quantity to the ultraviolet regions of momentum it is always clearer to use a cutoff regularization method such as Pauli-Villars. On the other hand, dimensional regularization will be much more practical when making more involved calculations, particularly in gauge theories. For this reason, this will be the method we will use for the rest of the lectures.

## Additional suggested readings


- *Gauge Theory of Elementary Particle Physics*, by T.-P. Cheng and L. -F. Li, Section 2.3
- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 10.2.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Section 8.1.2.