Lecture 22

Regularization I

Up to this point in the renormalization program we have classified theories based on their renormalizability and devised a way to isolate the divergences of a renormalizable theory by way of the counterterm method. Now it is time to deal with the divergences by regularizing the divergent integrals. As we discussed earlier, if in the regularization procedure we define an euclidean momentum cutoff Λ we expect that there will be a cutoff-dependent piece signifying the divergence, as well as a finite part that should be independent of Λ in the $\Lambda \to \infty$ limit.

Whatever the regularization method we choose, it must respect the symmetries of the theory. In a relativistic quantum field theory, these include Lorentz symmetry, as well as whatever other internal symmetries of the action (i.e. gauge and/or global symmetries). We will consider two regularization methods: *Pauli-Villars* (PV) and *Dimensional Regularization* (DR). Of course, the computation of any renormalized observable quantity should give the same answer in any regularization method. As we will see below, what will be different is the way to express the divergences in the counterterms. We will start with the Pauli-Villars method since it is more intuitive. Along the way we will collect some results that we will need also for DR. The DR technique is more widely used in practice since is less cumbersome in most applications.

22.1 Pauli-Villars Regularization

The idea behind PV regularization is very simple. The degree of divergence of a given integral is increased by positive powers of momenta. These come from the differential volume in momentum space and derivatives coming from the Feynman rules of the interactions. On the other hand, propagators introduce negative powers of momenta. For instance, let us consider again the one-loop contribution to the four-point function in the ϕ^4 theory as introduced in Lecture 20. In particular, let us focus on the s-channel diagram of Figure 20.5(a), depicted again below in Figure 22.1.

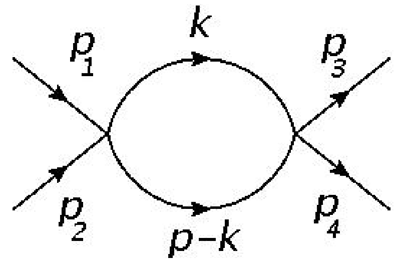

Figure 22.1: The s-channel one-loop correction to the four-point function $\Gamma(s,t,u)$ in $\lambda\phi^4$ theory. Here $p = p_1 + p_2 = p_3 + p_4$.

This contribution is given by

$$\Gamma(p^2) = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon} , \tag{22.1}$$

where $p = p_1 + p_2$, m is the scalar mass and λ the quartic coupling. As we discussed earlier, the integral in (22.1) is logarithmically divergent. The PV trick here consists in making the integral convergent by replacing one of the propagators as in

$$\begin{aligned} \frac{1}{k^2 - m^2 + i\epsilon} &\rightarrow \frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon} \\ &= \frac{m^2 - \Lambda^2}{(k^2 - m^2 + i\epsilon)(k^2 - \Lambda^2 + i\epsilon)} , \end{aligned} \tag{22.2}$$

where Λ is a mass parameter, and at the end of the calculation we take the limit $\Lambda \to \infty$ to recover the original theory. Of course, if we take the limit now we get the original propagator. But we want to take the limit after integration in order isolate the pieces that diverge with Λ. With the replacement in (22.2), now the contribution in (22.1) reads

$$\Gamma(p^2) = -\frac{\lambda^2 \Lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)(k^2 - \Lambda^2 + i\epsilon)\left[(p-k)^2 - m^2 + i\epsilon\right]} , \tag{22.3}$$

where we have already used the fact that $\Lambda \gg m$ in anticipation of taking the $\Lambda \to \infty$ limit. Clearly the integral in (22.3) is now finite since the replacement of the propagator in (22.2) resulted in a larger power of the momentum k in the denominator.

More generally, it will always be possible to make integrals appearing in loop calculations finite by replacing at least one propagator such as in

$$\frac{1}{k^2 - m^2 + i\epsilon} \to \frac{i}{k^2 - m^2 + i\epsilon} + \sum_i \frac{a_i}{k^2 - \Lambda_i^2 + i\epsilon} \,, \tag{22.4}$$

where the Λ_i's are mass parameters to be taken to ∞ at the end of the calculation, and the coefficients a_i are to be determined in each case. For instance, in (22.2) we only needed one PV term, with $a_1 = -1$ in order to reduce the degree of divergence of the integral enough to make it finite.

Going back to our case in (22.1), we can now Taylor-expand $\Gamma(p^2)$ about some value of the external momentum. Why do this ? We saw also in Lecture 20 that Taylor expansion of n-point functions allows us to separate the divergences in them in a few terms as with each derivative with respect to the external momentum the integrals become more convergent. For the case at hand, i.e. the four-point function, we saw that the second term in the expansion, with one derivative was already finite. Expanding about $p^2 = 0$ we write

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2) \,, \tag{22.5}$$

where

$$\Gamma(0) = -\frac{\lambda^2}{2}\Lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)^2(k^2 - \Lambda^2 + i\epsilon)} \,, \tag{22.6}$$

whereas

$$\begin{aligned} \tilde{\Gamma}(p^2) &= \Gamma(p^2) - \Gamma(0) \\ &= -\frac{\lambda^2}{2}\Lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2p\cdot k - p^2)}{(k^2 - m^2)^2\,(k^2 - \Lambda^2)\,[(p-k)^2 - m^2]} \,. \end{aligned} \tag{22.7}$$

From (22.6) we can see that $\Gamma(0)$ is divergent as $\Lambda \to \infty$. On the other hand, in this limit $\tilde{\Gamma}(p^2)$ is finite and given by

$$\tilde{\Gamma}(p^2) = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{(2p\cdot k - p^2)}{(k^2 - m^2 + i\epsilon)^2\,[(p-k)^2 - m^2 + i\epsilon]} \,. \tag{22.8}$$

So we have succeed in expressing the s-channel one-loop contribution to the four-point function as the sum of a divergent integral and a well defined and finite function of p^2, the external momentum. The same can be done with the other two contributions, the t and u channels. In order to continue further we will need to perform the four-dimensional integrals in (22.6) and (22.8). This requires we introduce two important steps: *Feynman parametrization* and *Wick rotation.*

22.2 Feynman Parametrization

To perform the integrals in the four-momentum k_μ we will start by using a trick due to Feynman to combine the factors in the denominator into just one factor. Let us start with the following simple case of two factors we call A and B. We can write them as

$$\frac{1}{A\,B} = \int_0^1 dx\, \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{1}{[xA+yB]^2}\ . \tag{22.9}$$

The first equality can be easily checked explicitly. The second one is trivial but it will help generalize for the case with more than two or three factors in the denominator. Here, x and y are called Feynman parameters. Let us do an example that will be of relevance for our integrals. We can write

$$\frac{1}{(k^2-m^2)(k-p)^2} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{1}{[x(k-p)^2 + y(k^2-m^2)]^2}\ . \tag{22.10}$$

Here k and p are four momenta and m some mass parameter or any constant with units of energy. We can rewrite the argument of the denominator as

$$x\,(k-p)^2 + y\,(k^2-m^2) = k^2 - 2x\,k\cdot p + x\,p^2 - (1-x)\,m^2\ , \tag{22.11}$$

where we have enforced the δ function, i.e. we used $y = 1-x$. But to actually perform the integration, we would like to complete the square so as to have a function of the square of the variable of integration. For this purpose, we define

$$\ell_\mu \equiv k_\mu - x\,p_\mu\ . \tag{22.12}$$

Then, we can write

$$\frac{1}{(k^2-m^2)(k-p)^2} = \int_0^1 dx\,\frac{1}{[\ell^2 - x^2\,p^2 + x\,p^2 - (1-x)\,m^2]^2}\ , \tag{22.13}$$

where now we see that the momentum to be integrated is ℓ_μ, which is fine since the shift (22.12) results in $d^4k = d^4\ell$. The resulting momentum integral will be now easy to perform. But before doing that, let us generalize the Feynman trick to more complicated cases, including the ones we have in (22.6) and (22.8).

For the integral in (22.8) we will need

$$\begin{aligned} \frac{1}{A\,B^2} &= -\frac{d}{dB}\left(\frac{1}{A\,B}\right) \\ &= -\frac{d}{dB}\int_0^1 dx\,\frac{1}{\left[x\,A+(1-x)\,B\right]^2}\ , \end{aligned} \tag{22.14}$$

where we are just using the derivative with respect to B in order to bring out its extra power in the denominator. The result is

$$\frac{1}{A\,B^2} = \int_0^1 dx\,\frac{2\,(1-x)}{\left[x\,A+(1-x)\,B\right]^3} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{2\,y}{\left[x\,A+y\,B\right]^3}\ . \tag{22.15}$$

In the second equality we again introduced the δ function for generalization purposes. In fact, for n powers of B it is easy to check that we can write

$$\frac{1}{A\,B^n} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{n\,y^{n-1}}{\left[x\,A+y\,B\right]^{n+1}}\ . \tag{22.16}$$

Also it is easy to prove (e.g. by induction starting with the $A\,B$ denominator) that when we have n denominators A_i we can write

$$\frac{1}{A_1\,A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\,\delta\left(\sum_i^n x_i-1\right)\,\frac{(n-1)!}{\left[x_1\,A_1+x_2\,A_2+\cdots+x_n\,A_n\right]^n}\ . \tag{22.17}$$

In this way, the Feynman parametrization trick allows us to always turn the product of denominators involving the integration variable into integrals over Feynman parameters but involving only one denominator in term of the shifted integration momentum, such as ℓ_μ in (22.12).

Now we can go back to the integral in (22.8). Using our result in (22.15), the factor of interest is written as

$$\frac{1}{(k^2-m^2+i\epsilon)^2\,[(p-k)^2-m^2+i\epsilon]} = \int_0^1 dx\,\frac{2\,(1-x)}{\left[x\,((p-k)^2-m^2)+(1-x)\,(k^2-m^2)\right]^3}$$

$$\equiv \int_0^1 dx\,\frac{2\,(1-x)}{D^3}\,, \tag{22.18}$$

where in the last equality we defined the argument of the denominator as D. We can easily check that it can be written as

$$D = k^2 - 2x\,k\cdot p + x\,p^2 - m^2\ . \tag{22.19}$$

In this way we can rewrite (22.8) as

$$\tilde{\Gamma}(p^2) = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{2\,(1-x)\,(2p\cdot k-p^2)}{D^3}\ . \tag{22.20}$$

Shifting the momentum being integrated by a constant just as in (22.12) we now have

$$D = \ell^2 + x(1-x)\,p^2 - m^2\ . \tag{22.21}$$

We should also used (22.12) in the numerator in (22.20). Since

$$2p\cdot k - p^2 = 2p\cdot\ell + 2x\,p^2 - p^2 = 2p\cdot\ell + (2x-1)\,p^2\ , \tag{22.22}$$

we now have

$$\tilde{\Gamma}(p^2) = \frac{\lambda^2}{2}\int_0^1 dx\,2\,(1-x)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{(2p\cdot\ell+(2x-1)\,p^2)}{\left[\ell^2-a^2+i\epsilon\right]^3}\ , \tag{22.23}$$

where we have restored the $i\epsilon$ that we had omitted for simplicity, and we defined

$$a^2 \equiv m^2 - x(1-x)\,p^2\ , \tag{22.24}$$

which is just a constant with respect to the momentum ℓ integration. We are now in a position to perform the integral in ℓ, once we go to euclidean four-dimensional space via a Wick rotation.

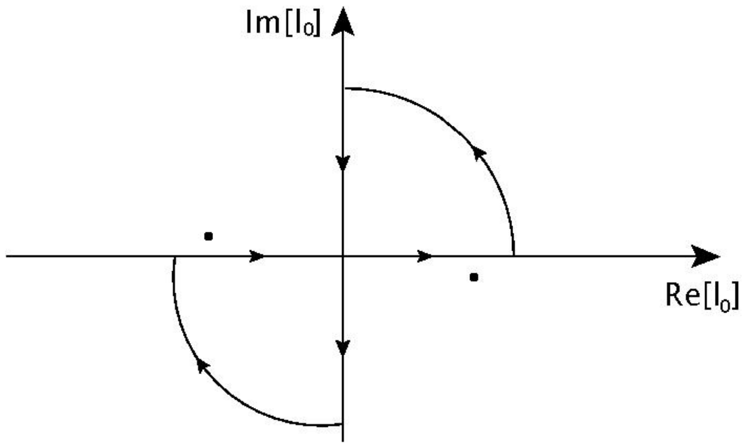

Figure 22.2: The ℓ_0 complex plane. The dots are the poles in (22.26). The contour excludes them leading to the vanishing of the integral. Thus, the integral along the real axis is equal to the one along the imaginary axis.

22.3 Wick Rotation

The argument in the denominator in (22.23) can be decomposed as

$$\begin{aligned} \ell^2 - a^2 + i\epsilon &= \ell_0^2 - \vec{\ell}^2 - a^2 + i\epsilon \\ &= \ell_0^2 - \left[\left(\vec{\ell}^2 + a^2 \right)^{1/2} - i\epsilon \right]^2 . \end{aligned} \tag{22.25}$$

Then, if we analytically continue to the complex ℓ_0 plane, we see that the integrand in (22.23) has two poles at:

$$\ell_0 = \begin{cases} \left(\vec{\ell}^2 + a^2 \right)^{1/2} - i\epsilon \\ \\ -\left(\vec{\ell}^2 + a^2 \right)^{1/2} + i\epsilon \end{cases} \tag{22.26}$$

Figure 22.2 shows a contour that excludes the poles. Then, Cauchy's theorem tells us that

$$\oint f(\ell_0)\, d\ell_0 = 0 \ , \tag{22.27}$$

with

$$f(\ell_0) \equiv \frac{1}{\left[\ell_0^2 - \left(\vec{\ell}^2 + a^2\right)^{1/2} - i\epsilon\right]^3} \,. \tag{22.28}$$

Furthermore, for $|\ell_0| \to \infty$ we have that $f(\ell_0) \to 0$. Then, the quarter circle parts of the contour will not contribute if at ∞. Then, (22.27) implies that

$$\int_{-\infty}^{\infty} f(\ell_0)\, d\ell_0 = \int_{-i\infty}^{i\infty} f(\ell_0)\, d\ell_0 \,. \tag{22.29}$$

In other words, the integral over the real part of ℓ_0, which is the integral we need to perform in (22.23), is the same as the integral over the imaginary part of ℓ_0 by virtue of Cauchy's theorem. A way to take advantage of this is by replacing ℓ_0 by an imaginary number as in

$$\ell_0 = i\ell_4 \tag{22.30}$$

Then we have that the integral over the imaginary part can be written as

$$\begin{aligned}\int_{-i\infty}^{i\infty} d\ell_0\, f(\ell_0) &= \int_{-\infty}^{\infty} i\, d\ell_4\, f(i\,\ell_4) \\ &= i \int_{-\infty}^{\infty} d\ell_4 \frac{(-1)}{\left[\ell_4^2 + \vec{\ell}^2 + a^2 + i\epsilon\right]^3} \,.\end{aligned} \tag{22.31}$$

What we have effectively achieved is to write the integral over the Minkowskian four-momentum ℓ_μ, as an integral over the euclidean four-momentum defined by

$$\ell_E^2 \equiv \ell_4^2 + \vec{\ell}^2 \,, \tag{22.32}$$

such that our integral is now over the euclidean four-momentum ℓ_E:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - a^2 + i\epsilon]^3} = -i \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{[\ell_E^2 + a^2 - i\epsilon]^3} \,. \tag{22.33}$$

This is the result of what we call a Wick rotation: take ℓ_0 in the integral on the left and change it by $i\ell_4$ everywhere. That is $\ell_0 \to i\ell_4$. But this is the same as going to euclidean space from Minkowski since

$$\ell_0^2 - \vec{\ell}^2 \to -\left(\ell_4^2 + \vec{\ell}^2\right) = -\ell_E^2 \ . \tag{22.34}$$

This is very good since we know how to do integrals in euclidean space.

Let us start with the angular part. We have

$$\int d^4\ell_E = \int d\ell_E\, \ell_E^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} \sin^2\chi\, d\chi = 2\pi^2 \int_0^{\infty} d\ell_E\, \ell_E^3 \ , \tag{22.35}$$

where we used the fact that in euclidean four dimensions we have two polar angles and one azimuthal. Then the integral in (22.33) is

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - a^2 + i\epsilon]^3} &= -i\frac{2\pi}{(2\pi)^4} \int_0^{\infty} d\ell_E \frac{\ell_E^3}{[\ell_E^2 + a^2 - i\epsilon]^3} \\ &= \frac{(-i)}{16\pi^2} \int_0^{\infty} d\ell_E^2 \frac{\ell_E^2}{[\ell_E^2 + a^2 - i\epsilon]^3} \ ,\end{aligned} \tag{22.36}$$

where in the last step we made use of

$$d\ell_E\, \ell_E^3 = \frac{1}{2}\, d\ell_E^2\, \ell_E^2 \ . \tag{22.37}$$

The integral in (22.36) can be performed by noticing that is just a *beta function*. These are defined by

$$B(m, n-m) \equiv \int_0^{\infty} \frac{t^{m-1}}{(1+t)^n}\, dt = \frac{\Gamma(m)\,\Gamma(n-m)}{\Gamma(n)} \ , \tag{22.38}$$

where in the last equality we express the beta function as a product of gamma functions. Then we see that

$$\begin{aligned}\int_0^{\infty} d\ell_E^2 \frac{\ell_E^2}{[\ell_E^2 + a^2]^3} &= \frac{1}{a^2} \int_0^{\infty} \left(d\ell_E^2/a^2\right) \frac{(\ell_E^2/a^2)}{[1 + \ell_E^2/a^2]^3} \\ &= \frac{1}{a^2}\, B(2, 3-2) = \frac{1}{a^2} \frac{\Gamma(2)\,\Gamma(3-2)}{\Gamma(3)} = \frac{1}{2\,a^2} \ .\end{aligned} \tag{22.39}$$

So we are done! But before we put it all together and write the answer for $\tilde{\Gamma}(p^2)$ we notice that in the expression for it (22.23) the term in the integrand that is linear in ℓ_μ will not contribute to the result since it corresponds to an odd power of ℓ multiplied by an even function and integrated over all values of ℓ. So its contribution is zero. With this in mind we can rewrite (22.23) as

$$\tilde{\Gamma}(p^2) = \lambda^2 \int_0^1 dx\,(1-x) \int \frac{d^4\ell}{(2\pi)^4}\, \frac{(2x-1)\,p^2}{\left[\ell^2 - a^2 + i\epsilon\right]^3}\,, \tag{22.40}$$

Then using the results from (22.36) and (22.39) we have that (22.40) can be expressed as

$$\boxed{\tilde{\Gamma}(p^2) = -i\,\frac{\lambda^2}{32\pi^2}\,p^2 \int_0^1 dx\,\frac{(1-x)\,(2x-1)}{m^2 - x(1-x)\,p^2 - i\epsilon}}\,, \tag{22.41}$$

where we made use of (22.24) and we restored the $-i\epsilon$. It is important to realize that this is a contribution to the four-point correlation function that introduces a *physical* momentum dependence and that it does not depend on the regulator or the regularization procedure. Together with the other two contributions to one loop described in Lecture 20, it constitutes a calculation of a physical effect: the dependence with external momentum of the four-point amplitude.

It is interesting to obtain the final functional form by actually performing the final integral over the Feynman parameter x. The most general form of the result is

$$\tilde{\Gamma}(s) = \begin{cases} \frac{i\lambda^2}{32\pi^2}\left(2 - 2\sqrt{\frac{4m^2-s}{s}}\,\arctan\left[\sqrt{\frac{s}{4m^2-s}}\right]\right); & 0 < s < 4m^2 \\[2ex] \frac{i\lambda^2}{32\pi^2}\left(2 + \sqrt{\frac{s-4m^2}{s}}\,\ln\left[\frac{\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}+\sqrt{s-4m^2}}\right] + i\pi\right); & s > 4m^2 \\[2ex] \frac{i\lambda^2}{32\pi^2}\left(2 + \sqrt{\frac{4m^2-s}{|s|}}\,\ln\left[\frac{\sqrt{4m^2-s}-\sqrt{|s|}}{\sqrt{4m^2-s}+\sqrt{|s|}}\right]\right); & s < 0 \end{cases} \tag{22.42}$$

Some comments about this result are in order. The physical region corresponds to $s > 4m^2$. Here we see that the four-point function, and therefore the associated amplitude, acquires an imaginary part. This happens every time the external momentum is large enough to produce the virtual particles in the loop as real states. When that is the case, as it is here, the imaginary part corresponds to the part of the amplitude that is actually the product of two tree-level amplitudes: that of producing the intermediate state on-shell times that of these states rescattering to give the final state. We will see this again later in the course. On the other hand, the unphysical regions, such as $s < 0$ are of interest since the four-point function is an analytical function of the Mandelstam variables, so we can analytically continue it to these regions of phase space.

Finally, we need to compute the divergent piece of the four-point function, $\Gamma(0)$. Recall that from (22.6) we have

$$\begin{aligned}
\Gamma(0) &= -\frac{\lambda^2}{2}\Lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\epsilon)^2(k^2-\Lambda^2+i\epsilon)} \\
&= -\frac{\lambda^2}{2}\Lambda^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{2\,(1-x)}{\left[x\,(k^2-\Lambda^2)+(1-x)\,(k^2-m^2)\right]^3} \,, \qquad (22.43)
\end{aligned}$$

where in the last step we used (22.15). Defining the argument of the denominator as

$$D \equiv k^2 - x\,\Lambda^2 - (1-x)\,m^2 = k^2 - a^2 \,, \qquad (22.44)$$

and $a^2 = \Lambda^2 + (1-x)\,m^2$, we have

$$\begin{aligned}
\Gamma(0) &= -\frac{\lambda^2\,\Lambda^2}{16\pi^4} \int_0^1 dx\,(1-x) \int d^4k \frac{1}{\left[k^2-a^2+i\epsilon\right]^3} \\
&= -\frac{\lambda^2\,\Lambda^2}{16\pi^4} \int_0^1 dx\,(1-x) \int_0^\infty dk_E^2 \frac{k_E^2\,(1/2)\,(2\pi^2)}{(-1)\left[k_E^2+a^2-i\epsilon\right]^3} \\
&= +i\frac{\lambda^2\,\Lambda^2}{16\pi^2} \int_0^1 dx\,(1-x) \int_0^\infty dk_E^2 \frac{k_E^2}{\left[k_E^2+a^2-i\epsilon\right]^3} \,. \qquad (22.45)
\end{aligned}$$

The second line in (22.45) is just the Wick rotation with the angular integral resulting in the factor of $2\pi^2$ just as in (22.35). The integral in k_E^2 in the last line is just the same as the one in (22.39). So we obtain

$$\begin{aligned}
\Gamma(0) &= i\frac{\lambda^2\,\Lambda^2}{16\pi^2} \int_0^1 dx \frac{(1-x)}{2\,(a^2-i\epsilon)} \\
&= i\frac{\lambda^2\,\Lambda^2}{32\pi^2} \int_0^1 dx \frac{(1-x)}{x\,\Lambda^2+(1-x)\,m^2-i\epsilon} \,. \qquad (22.46)
\end{aligned}$$

Finally, it is possible to integrate over the Feynman variable x resulting in

$$\Gamma(0) = i\frac{\lambda^2}{32\pi^2}\frac{1}{(1-m^2/\Lambda^2)}\left[\frac{1}{(1-m^2/\Lambda^2)}\ln\left(\frac{\Lambda^2}{m^2}\right)-1\right] \,. \qquad (22.47)$$

This expression for $\Gamma(0)$ clearly exhibits the logarithmic divergence that was anticipated. Since $\Lambda^2 \gg m^2$, the expression can be further simplified to

$$\boxed{\Gamma(0) \simeq i\frac{\lambda^2}{32\pi^2}\ \ln\left(\frac{\Lambda^2}{m^2}\right)}\ . \tag{22.48}$$

With (22.48) and (22.41) we are now in a position to apply the procedure of the previous lecture to fix the value of the counterterm $\delta\lambda$ associated to the four-point function. In particular, following Section 21.3 of Lecture 21, we recall that the amplitude has the form

$$i\mathcal{A}(p_1, p_2 \to p_3, p_4) = -i\lambda + \Gamma(s) + \Gamma(t) + \Gamma(u) - i\delta\lambda\ , \tag{22.49}$$

where the last term is the counterterm. Then using the renormalization point at $s_0 = 4m^2$, $t_0 = u_0 = 0$, we imposed the renormalization condition

$$i\mathcal{A}(s_0, t_0, u_0) = -i\lambda\ , \tag{22.50}$$

which is the actual definition of the renormalized coupling λ. Then, for the counterterm we obtained

$$\delta\lambda = -i\left[\Gamma(4m^2) + 2\Gamma(0)\right] = -i\left[3\Gamma(0) + \tilde{\Gamma}(4m^2)\right]\ . \tag{22.51}$$

Using (22.42), we get that

$$\tilde{\Gamma}(4m^2) = \frac{i\lambda^2}{32\pi^2}\left(2 + i\pi\right)\ , \tag{22.52}$$

which results in

$$\boxed{\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 2 - i\pi\right]}\ . \tag{22.53}$$

This expression for the four-point function counterterm $\delta\lambda$ has the expected logarithmic divergence. However, its detailed form will depend on the regularization procedure. We will compute $\delta\lambda$ also in another regularization method, dimensional regularization, in order to compare the two results.

22.4 Regularization of the Two-point Function

We will now compute the counterterms of the two-point function to leading order corrections in the couling λ. We start by remembering the expression of the two-point function to one loop order. This is

$$-i\Sigma(p^2) = -i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} + i(\delta z\, p^2 - \delta m^2)\ , \tag{22.54}$$

where we have included the appropriate counterterms for the field wave-function δz, and for the mass δm^2. As we know from earlier discussions, the integral diverges quadratically. We want to use Pauli-Villars regularization. Then we need to increase the power of momentum in the denominator such that for large (euclidean) momentum we have a finite integral. That is for $|k|^2 \to \infty$ we need

$$\int \frac{d^4k}{k^2} \to \int \frac{d^4k}{k^6}\ , \tag{22.55}$$

as a result of the Pauli-Villars procedure. This can be achieved by adding two massive propagators. Thus, we replace the propagator in (22.54) following the prescription

$$\frac{1}{k^2-m^2} \to \frac{1}{k^2-m^2} + \frac{a_1}{k^2-\Lambda_1^2} + \frac{a_2}{k^2-\Lambda_2^2}\ , \tag{22.56}$$

where a_1 and a_2 are constants to be determined, and in principle we could have different fictitious masses Λ_1 and Λ_2. Just doing the algebra we obtain

$$\frac{1}{k^2-m^2} \to \frac{(k^2-\Lambda_1^2)(k^2-\Lambda_2^2) + a_1(k^2-m^2)(k^2-\Lambda_2^2) + a_2(k^2-m^2)(k^2-\Lambda_1^2)}{(k^2-m^2)(k^2-\Lambda_1^2)(k^2-\Lambda_2^2)}\ . \tag{22.57}$$

We will now fix a_1 and a_2 by imposing the UV behavior (22.55) so that the integral is finite. For this purpose, all the non-zero powers of momentum in the numerator in (22.57) should vanish to get the $1/k^6$ suppression. First, we impose that in (22.57) the k^4 power contributions vanish. This happens if

$$\boxed{1 + a_1 + a_2 = 0}. \tag{22.58}$$

Next, we impose that the k^2 terms vanish too. This leads to

$$\boxed{-\Lambda_1^2 - \Lambda_2^2 - a_1\Lambda_2^2 - a_2\Lambda_1^2 - (a_1+a_2)m^2 = 0}. \tag{22.59}$$

Solving the system of equations in (22.58) and (22.59) we arrive at

$$\boxed{a_1 = \frac{m^2 - \Lambda_2^2}{\Lambda_2^2 - \Lambda_1^2}\ , \qquad a_2 = \frac{\Lambda_1^2 - m^2}{\Lambda_2^2 - \Lambda_1^2}}\ . \tag{22.60}$$

Replacing (22.60) into (22.57) we obtain

$$\frac{1}{k^2 - m^2} \to \frac{(\Lambda_1^2 - m^2)(\Lambda_2^2 - m^2)}{(k^2 - m^2)(k^2 - \Lambda_1^2)(k^2 - \Lambda_2^2)}\ . \tag{22.61}$$

We will be interested in taking the limit $\Lambda \to \infty$, but we will keep it at a finite cutoff Λ for now. However, we can take $\Lambda_1 = \Lambda_2 = \Lambda$ for this purpose. Then we have that (22.61) turns into

$$\frac{1}{k^2 - m^2} \to \frac{\Lambda^4}{(k^2 - m^2)(k^2 - \Lambda^2)^2}\ . \tag{22.62}$$

We are now ready to perform the integral in (22.54). With the replacement in (22.62) we have

$$\begin{aligned} I &\equiv \frac{(-i\lambda)}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\ &= \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\Lambda^4}{(k^2 - m^2)\,(k^2 - \Lambda^2)^2}\ . \end{aligned} \tag{22.63}$$

Once we have this form, we can follow the same steps that we used for the four-point function. First, we implement the <u>Feynman parametrization</u> by using

$$\frac{1}{(k^2 - m^2)\,(k^2 - \Lambda^2)^2} = \int_0^1 dx\, \frac{2(1-x)}{\left[x(k^2 - m^2) + (1-x)(k^2 - \Lambda^2)\right]^3}\ . \tag{22.64}$$

Changing the integration variable as

$$y \equiv 1 - x\ , \tag{22.65}$$

and rewriting the integral in the Feynman parameter , we now have

$$\frac{1}{(k^2-m^2)\,(k^2-\Lambda^2)^2} = -\int_0^1 dy\,\frac{2y}{[k^2-y\Lambda^2+(1-y)m^2]^3}\ . \tag{22.66}$$

Then, the integral in (22.63) can be written as

$$I = -\frac{\lambda}{2}\,\Lambda^4\int_0^1 dy\,2y\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-a^2+i\epsilon]^3}\ , \tag{22.67}$$

where we have defined

$$a^2 \equiv y\Lambda^2+(1-y)m^2\ , \tag{22.68}$$

and we have restored the $i\epsilon$.

The momentum integral in (22.67) is the one we performed before for the case of the four-point function. First, we Wick-rotate

$$k_0 \to ik_E\ , \tag{22.69}$$

to obtain

$$\begin{aligned} I &= i\frac{\lambda}{2}\,\frac{\Lambda^2}{16\pi^4}\int_0^1 dy\int\frac{d^4k_E}{[k_E^2+a^2-i\epsilon]^3}, \\ &= i\frac{\lambda}{2}\,\frac{\Lambda^2}{16\pi^4}\int_0^1 dy\,\frac{\pi^2}{2(a^2-i\epsilon)} \end{aligned} \tag{22.70}$$

where in the last step we performed the integral over the euclidean momentum using beta and gamma functions. The remaining integral in the Feynman parameter y can be easily performed. We have

$$I = i\frac{\lambda\,\Lambda^4}{32\pi^2}\int_0^1 dy\,\frac{y}{y\Lambda^2+(1-y)m^2}\ . \tag{22.71}$$

Keeping the leading terms in the cutoff Λ we obtain

$$I \simeq i\frac{\lambda}{32\pi^2}\,\left\{\Lambda^2-m^2\ln\left(\frac{\Lambda^2}{m^2}\right)\right\}\ , \tag{22.72}$$

where we threw away terms that go to zero in the $\Lambda \to \infty$ limit. We see that, just as expected, the integral is quadratically divergent, and there is a milder logarithmic divergence. Finally, we use this result to compute the two-point function counterterms appearing in (22.54) to this order. Since the one-loop integral in (22.72) has no external momentum dependence, we then conclude that, at least at this order, $\delta z = 0$. We can then use (22.54) and (22.72) to extract the mass counterterm. Since we have

$$-i\Sigma(p^2) = I - i\delta m^2 \ , \tag{22.73}$$

there is no p^2 dependence to this order in perturbation theory, and as a result the renormalization condition defining the counterterm δm^2 can be chosen either at $p^2 = 0$ or $p^2 = m^2$ without distinction. In other words

$$-i\Sigma(0) = -i\Sigma(m^2) = 0 \ , \tag{22.74}$$

results in

$$\boxed{\delta m^2 = \frac{\lambda}{32\pi^2}\left\{\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right\}} \ , \tag{22.75}$$

up to corrections ot order λ^2. Thus, together with

$$\boxed{\begin{aligned} &\delta z = 0 \ , \\ &\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left\{\ln\left(\frac{\Lambda^2}{m^2}\right) - (2 + i\pi)\right\} \ , \end{aligned}}$$

also obtained up to λ^2 corrections, we complete our knowledge of the counterterms of the theory. With these counterterms we can compute any amplitude in the renormalized theory with lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta z\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \ . \tag{22.76}$$

Here, the field ϕ, the mass and the coupling λ are renormalized parameters. All amplitudes computed with this lagrangian will be finite and compatible with the renormalization conditions. Thus, we conclude the program of renormalization by counterterms, where these were computed by the Pauli-Villars method. Next we will introduce another regularization method that, although a bit more involved and less intuitive at first, will be more powerful when used in most applications.

Additional suggested readings

- *Gauge Theory of Elementary Particle Physics*, by T.-P. Cheng and L. -F. Li, Section 2.3
- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 10.2.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Section 8.2.