Lecture 21

Renormalization by Counterterms

Starting from the lagrangian for a real scalar field with quartic self-interactions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \ , \tag{21.1}$$

with unrenormalized parameters m_0^2, λ_0 and the unrenormalized field ϕ_0, we defined the renormalized parameters m^2, λ and ϕ. First, we start by the field, just as we did in the previous lecture.

$$\phi = Z_\phi^{-1/2}\,\phi_0 \ . \tag{21.2}$$

Then, we rewrite (21.1) replacing the renormalized field ϕ for ϕ_0 to obtain

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2 Z_\phi\phi^2 - \frac{\lambda_0}{4!}Z_\phi^2\phi^4 \ , \tag{21.3}$$

We now define

$$\begin{aligned}
\delta Z_\phi &\equiv Z_\phi - 1 \\
\delta m^2 &\equiv m_0^2 Z_\phi - m^2 \\
\delta\lambda &\equiv \lambda_0 Z_\phi^2 - \lambda \ ,
\end{aligned} \tag{21.4}$$

which we can rewrite in a more convenient way as

$$
\begin{aligned}
Z_\phi &= 1 + \delta Z_\phi \\
m_0^2 Z_\phi &= m^2 + \delta m^2 \\
\lambda_0 Z_\phi^2 &= \lambda + \delta\lambda \ .
\end{aligned}
\tag{21.5}
$$

Replacing (21.5) in (21.3) we obtain

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\
&+ \frac{1}{2}\delta Z_\phi\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \ ,
\end{aligned}
\tag{21.6}
$$

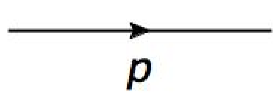
where we see that the first line is the renormalized lagrangian whereas the second line is what we will call the counterterms. These new terms will result in new Feynman rules for the theory and will cancel divergencies in the renormalized theory. We have seen (see lecture 19) that for this theory the degree of divergence of diagrams is given by

$$
D = 4 - \sum_f E_f(s_f + 1)
\tag{21.7}
$$

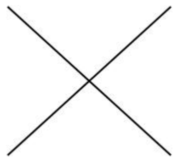
where E_f is the number of external lines of the field type f in the diagram and here $s_f = 0$ for a scalar field. This meant that there are divergences in the two-point function ($E_f = 2 \Rightarrow D = 2$) and in the four-point function ($E_f = 4 \Rightarrow D = 0$). The divergences in the two-point function affect the terms in $\mathcal{L}$ quadratic in the fields and there will be cancelled by the counterterms δZ_ϕ and δm^2, whereas the ones in the four-point function impact the quartic term and are canceled by $\delta\lambda$. The cancelation takes place at a given order in the perturbative expansion in the coupling constant λ. In order to define the physical parameters we need to impose renormalization conditions. To compute a given process up to some order in perturbation theory we need to use the Feynman rules that include the counterterms. These new contributions will ensure that the cancelation takes place in every process.

21.1 Counterterm Feynman Rules

The Feynman rules of the theory in terms of renormalized parameters are shown below, and derived from the first line in (21.6).

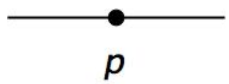


$$
\frac{i}{p^2 - m^2 + i\epsilon} \ ,
$$

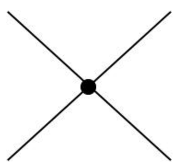


$$-i\lambda \ ,$$

In addition, we now need to derive new Feynman rules from the second line. This results in



$$i\left(\delta Z_\phi p^2 - \delta m^2\right) \ , \tag{21.8}$$



$$-i\delta\lambda \ , \tag{21.9}$$

where the dots indicate the insertion of the counterterm. To understand the form of the counterterm for the two-point function we should imagine inserting it as one more 1PI contribution to $-i\Sigma(p^2)$ in the summed propagator, as we did in lecture 20. With the form (21.8) the propagator now would be

$$\frac{i}{p^2 - m^2 - \Sigma_\ell(p^2) + \delta Z_\phi\, p^2 - \delta m^2} \ , \tag{21.10}$$

where $-i\Sigma_\ell(p^2)$ is the sum of the actual loop contributions to the two-point function. Notice that since the mass squared in the propagator is already the renormalized mass, the divergences in $-i\Sigma_\ell(p^2)$ will now be canceled exclusively by the counterterms δZ_ϕ and δm^2. To implement the program of renormalization by counterterms, we compute any desired amplitude up to the desired order in λ, including all the counterterms. Divergent integrals must be regulated, i.e. expressed in terms of an appropriate regulator that respects the symmetries of the theory. In the next lectures we will specify regularization procedures. But the regulator is typically either an euclidean momentum cut off Λ, or some other parameter that exposes the divergences in some limit. The answer of the calculation initially depends on the couterterms δZ_ϕ, δm^2 and $\delta\lambda$. These are fixed by imposing *renormalization conditions* that result in the cancellation of divergences. The resulting expression is then independent of the regulator. This procedure removes all divergences in a renormalizable theory.

21.2 Fixing δZ_ϕ and δm^2

These counterterms are fixed by the renormalization of the two-point function. The 1PI diagrams that need to be summed in order to obtain the propagator (21.10) now include the counterterm contribution, as shown in Figure 21.1, where we show the 1PI up to $\mathcal{O}(\lambda)$. In addition to the one-loop diagram we now have to consider the couterterm contribution to the two-point function as in (21.8) . The sum of the two diagrams is

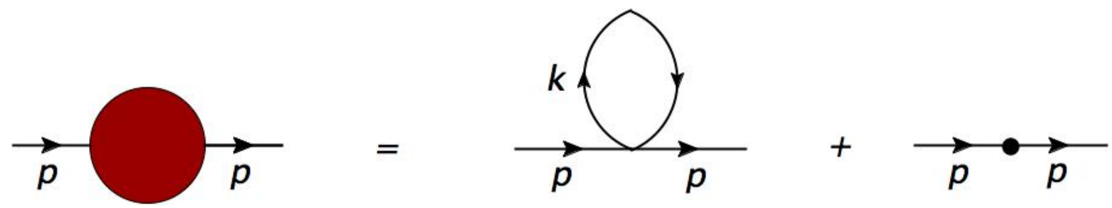


Figure 21.1: The 1PI diagrams contributing to the two-point function to $\mathcal{O}(\lambda)$. The last diagram corresponds to the counterterm in (21.8).

$$-i\Sigma(p^2) = \frac{(-i\lambda)}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon} + i\left(\delta Z_\phi p^2 - \delta m^2\right)\ . \qquad (21.11)$$

We will impose the renormalization conditions on the propagators

$$\Delta_F(p) = \frac{i}{p^2-m^2-\Sigma(p^2)}\ , \qquad (21.12)$$

we now we use the renormalized mass parameter m^2 from the lagrangian in (21.6) and $\Sigma(p^2)$ has the expansion

$$\Sigma(p^2) = \Sigma(m^2) + (p^2-m^2)\,\Sigma'(m^2) + \tilde{\Sigma}(p^2)\ , \qquad (21.13)$$

where the first two terms are divergent, but the last is not. Now, the renormalization conditions are a little different than before because here we are adding the contributions of the loop plus those of the counterterms and ger the *renormalized* propagator. This means that the renormalization condition now should leave m^2 as the pole *and* the residue should be unity times i, since the field is already renormalized. This translates into the conditions

$$\Sigma(m^2) = 0, \qquad\qquad \Sigma'(m^2) = 0, \qquad (21.14)$$

with the first condition ensuring that m^2 is the pole of the propagator, whereas the second one leads to the desired residue of i. We can see from (21.11) that, since the loop integral does not contain any p^2 dependence, the second renormalization condition in (21.14) leads to $\delta Z_\phi = 0$. However, this is only the case at this order in λ. In fact, going to $\mathcal{O}(\lambda^2)$ there will be such dependence in the integral, leading to the more accurate statement

$$\boxed{\delta Z_\phi = 0 + \mathcal{O}(\lambda^2)}\,. \tag{21.15}$$

Finally, we may use the first condition in (21.14) in (21.11) to obtain the mass squared counterterm

$$\boxed{\delta m^2 = -\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4)}\,\frac{i}{k^2 - m^2 + i\epsilon}}\,. \tag{21.16}$$

To actually compute δm^2 we will need to regulate the integral above. We will do this in detail in the nexttwo lectures. In any case, the answer will not depend on the details of the regularization procedure.

21.3 Fixing $\delta\lambda$ through the Four-point Function

The renormalization of the four-point function leads to the fixing of the coupling counterterm $\delta\lambda$. In this case the first loop corrections will introduce a momentum dependence absent at leading order. Let us consider a scattering process in the ϕ^4 theory up to one loop. The relevant diagrams are shown in Figure 21.2.

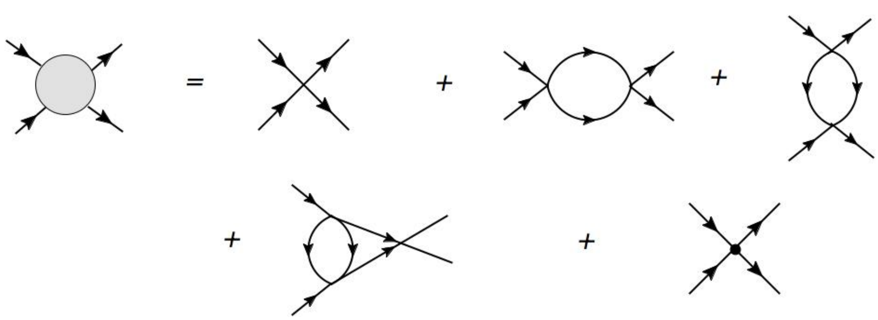


Figure 21.2: The diagrams contributing to the four-point amplitude. The leading order, i.e. $\mathcal{O}(\lambda)$ diagram is followed by the three possible $\mathcal{O}(\lambda^2)$ 1PI diagrams. The last diagram is the counterterm $\delta\lambda$.

The amplitude for scattering two scalars of momenta p_1 and p_2 into two scalars of momenta p_3 and p_4 is

$$i\mathcal{A}(p_1, p_2 \to p_3, p_4) = -i\lambda + \Gamma(s) + \Gamma(t) + \Gamma(u) - i\delta\lambda \ , \tag{21.17}$$

where the Mandelstam variables are $s = (p_1 + p_2)^2$, $t = (p_3 - p_1)^2$ and $u = (p_4 - p_1)^2$. Since the loop diagrams introduce kinematic dependence, we need once again to choose a point in order to impose the renormalization condition on the four-point function. This time we choose the zero-momentum condition, i.e.

$$s_0 = 4m^2, \qquad t_0 = 0, \qquad u_0 = 0 \ , \tag{21.18}$$

which corresponds to $p_1 = p_2 = (m, \mathbf{0})$. We then impose the renormalization condition

$$i\mathcal{A}(s_0, t_0, u_0) = -i\lambda \ , \tag{21.19}$$

which results in

$$\boxed{\delta_\lambda = -i\left(\Gamma(4m^2) + 2\Gamma(0)\right)} \ . \tag{21.20}$$

We can then rewrite the amplitude as

$$i\mathcal{A}(s, t, u) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \ , \tag{21.21}$$

where the $\tilde{\Gamma}$'s are finite and satisfy $\tilde{\Gamma}(s_0) = \tilde{\Gamma}(t_0) = \tilde{\Gamma}(u_0) = 0$. The amplitude in (21.21) is expressed in terms of the renormalized coupling λ and it has a well defined kinematic dependence acquired at order λ^2 through the finite parts of the loop diagrams. In the next two lectures we will develop methods to regulate the divergent integrals.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Section 10.2.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Section 8.2.