Lecture 20

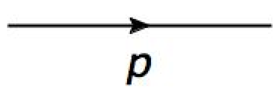
The Basis of the Renormalization Program

In order to introduce the main concepts involved in the renormalization process, we start by studying the correlation functions of a simple theory and derive the necessary elements to redefine them. Later on, we will develop a formalism that will incorporate all these steps.

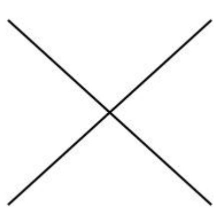
Let us consider the lagrangian of a real scalar field with ϕ^4 self-interactions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \ , \tag{20.1}$$

where the subscript 0 in the field, the mass and the coupling denotes the parameters of the theory before renormalization. The Feynman rules for this theory are the usual, in particular for the propagator and for the vertex we have



$$\frac{i}{p^2 - m_0^2 + i\epsilon} \ ,$$



$$-i\lambda_0 \ ,$$

in terms of the unrenormalized mass and coupling m_0 and λ_0. To account for the quantum corrections, we need to consider the one-particle irreducible (1PI) diagrams. These are diagrams that cannot be split into two by just cutting an internal line. Examples of 1PI diagrams are shown in Figure 20.1. They constitute the building blocks for quantum corrections. On the other hand, the examples of Figure 20.2 are one-particle *reducible* diagrams. They can be thought of as corrections to external legs, or in the case of the two-point function as repetitions of the 1PI building blocks.

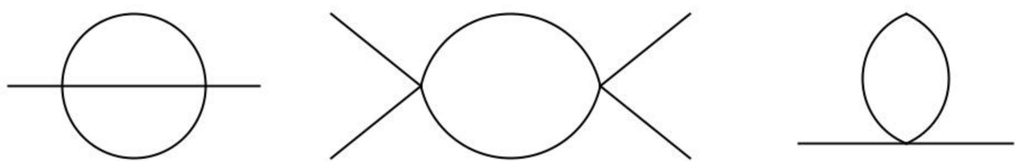


Figure 20.1: Examples of one-particle irreducible diagrams in loop corrections.

We will define 1PI correlation functions as those obtained from exclusively considering amputated 1PI diagrams[1]. We will denote the 1PI n-point correlation function in momentum space as

$$\Gamma^{(n)}(p_1, \ldots, p_n) \ ,$$

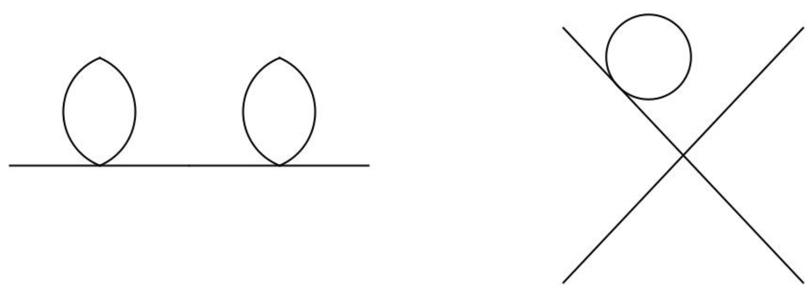


Figure 20.2: Examples of one-particle *reducible* diagrams in loop corrections.

The reducible diagrams can always be obtained from the product of 1PI diagrams without having to perform additional integrals. This means that the only divergences we have to worry about are those in the 1PI diagrams. Once these are absorbed in the renormalization procedure, there will be no more divergences in the reducible digrams either.

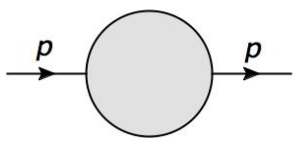
[1]Since we are computing the momentum-space correlation functions we must strip them from the external propagators.

20.1 Separation of Divergences

We will see how to isolate the divergences of the theory. Let us consider first the two-point function. In momentum space we denote this by

$$\Delta_F(p) = \int d^4x e^{ip\cdot x} \langle 0|T\phi_0(x)\phi_0(0)|0\rangle \ , \tag{20.2}$$

that is, this is the unrenormalized momentum-space propagator. We group all the relevant 1PI diagrams in



$$\equiv -i\Sigma(p^2) \ ,$$

where the blob includes all 1PI diagrams contributing to the desired order in perturbation theory. The total propagator requires that we sum over all the possible products of 1PI insertions as in Figure 20.3 below.

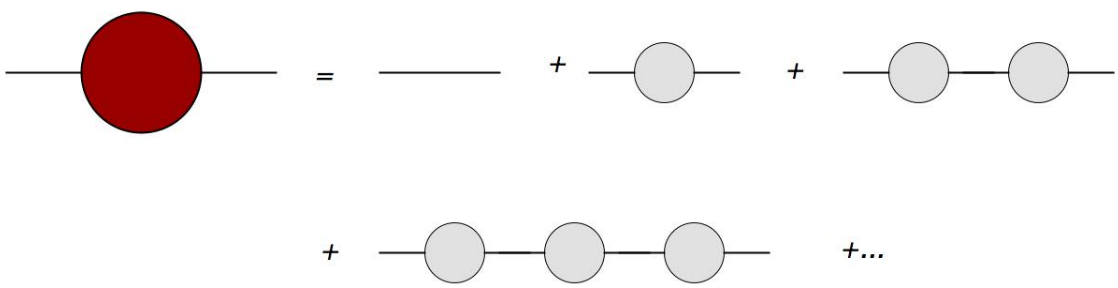


Figure 20.3: Insertion of the 1PI contributing to the two-point function, resulting in a geometric series.

The large blob denotes the full propagator to the desired order. This results in

$$\begin{aligned}
\Delta_F(p) &= \frac{i}{p^2 - m_0^2 + i\epsilon} + \frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m_0^2 + i\epsilon} \\
&+ \frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m_0^2 + i\epsilon} + \dots \\
&= \frac{i}{p^2 - m_0^2 + i\epsilon}\left\{\frac{1}{1 + i\Sigma(p^2)\frac{i}{p^2 - m_0^2 + i\epsilon}}\right\} \ ,
\end{aligned} \tag{20.3}$$

where in the last equality we summed the series under the assumption that $\Sigma(p^2)$ is "small". Thus the corrected propagator can now be written as

$$\boxed{\Delta_F(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon}} . \tag{20.4}$$

We can begin to see in (20.4) how is it that the corrections coming from the 1PI diagrams in $\Sigma(p^2)$ will result in a shift of the mass squared parameter, m_0^2, although as we will see in more detail later, this will not be the only paramater shift in the propagator. The function $\Sigma(p^2)$ shifts the pole of the propagator renormalizing the mass parameter, as well as the residue at the pole, resulting in a redefinition of the field itself. But before we do this, we notice that $\Sigma(p^2)$ is divergent. To leading order in λ_0 it is given by the diagram in Figure 20.4.

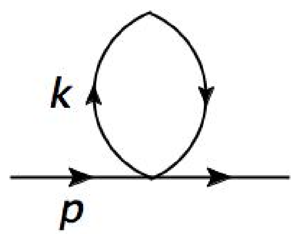


Figure 20.4: 1PI contribution to $-i\Sigma(p^2)$ up to order λ_0.

This results in

$$-i\Sigma(p^2) = \frac{-i\lambda_0}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} , \tag{20.5}$$

where the factor of 2 in the denominator accounts for the symmetry of the diagram. As we have seen before, this is quadratically divergent in the UV. It is possible to Taylor expand $-i\Sigma(p^2)$ about an arbitrary point and show that the divergences concentrate in the first few terms in the expansion. This will allow us to separate clearly the divergent and the convergent pieces of the two-point function, and in general of any correlation function. However, since to this order in λ_0 there is no p^2 dependence in $-i\Sigma(p^2)$ it is not obvious how to show this. To understand the separation of the divergences it is more convenient to study the four-point function.

The one-loop contributions to the four-point function are depicted in Figure 20.5. We will concentrate on the diagram (a) since the discussion is the same for all three with only minor kinematic differences. Using the Feynman rules for the ϕ^4 theory we obtain its contribution as

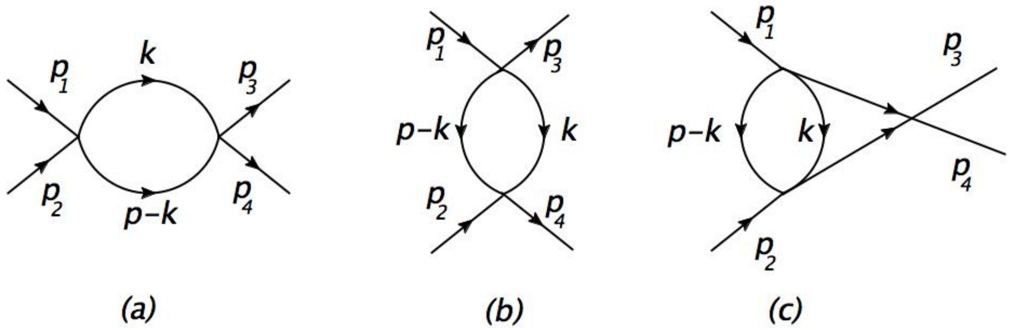


Figure 20.5: The 1PI one-loop diagrams contributing to the four-point function. In (a) $p = p_1 + p_2$. For (b) we have $p = p_1 - p_3$. Finally, for (c) $p = p_1 - p_4$.

$$\Gamma_{(a)}(p^2) = \frac{(-i\lambda_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \frac{i}{(p-k)^2 - m_0^2 + i\epsilon} \, , \tag{20.6}$$

where we used $p \equiv p_1 + p_2$. Since for this diagram we have $p^2 = s$, we have $\Gamma_{(a)}(p^2) = \Gamma(s)$. As noticed earlier, this integral is logarithmically divergent in the UV. Clearly for Figure 20.5-(b) we will have the same expression as in (20.6) with the only difference that $p = p_1 - p_3$. Thus, $\Gamma_{(b)}(p^2) = \Gamma(t)$. Finally, we have that for the diagram in (c) the result is the same function in (20.6) but now as function of $p^2 = (p_1 - p_4)^2 = u$, i.e. $\Gamma_{(c)}(p^2) = \Gamma(u)$. Then, the three contributions are given by the same function but with different argument. In particular the three of them have the same divergent structure.

In order to separate the divergences, we make the observation that taking derivatives of the correlation functions with respect to external momenta decreases the degree of divergence. We can see this readily in (20.6). We first notice that

$$\frac{\partial \Gamma(p^2)}{\partial p_\mu} = \frac{\partial \Gamma(p^2)}{\partial p^2} \frac{\partial p^2}{\partial p_\mu} = \frac{\partial \Gamma(p^2)}{\partial p^2} \, 2p^\mu \, , \tag{20.7}$$

from which we arrive at

$$\frac{\partial \Gamma(p^2)}{\partial p^2} = \frac{1}{2p^2} \, p_\mu \, \frac{\partial \Gamma(p^2)}{\partial p_\mu} \, . \tag{20.8}$$

Applying this expression to (20.6) we have

$$\frac{\partial \Gamma(p^2)}{\partial p^2} = \frac{\lambda_0^2}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)\cdot p}{k^2 - m_0^2 + i\epsilon} \frac{1}{\left[(p-k)^2 - m_0^2 + i\epsilon\right]^2} \, . \tag{20.9}$$

We see that the integral in (20.9) is already convergent in the UV. Additional derivatives will only result in even more rapidly convergent integrals. As a result we can Taylor expand $\Gamma(p^2)$ in order to isolate the divergent term. Expanding around $p^2 = 0$

$$\Gamma(p^2) = a_0 + a_1 p^2 + \cdots + \frac{1}{n!} a_n (p^2)^n + \ldots \ , \tag{20.10}$$

where the coefficients are given by

$$a_n = \left. \frac{\partial^n \Gamma(p^2)}{\partial (p^2)^n} \right|_{p^2=0} \ , \tag{20.11}$$

As we have seen above, a_0 diverges logarithmically, whereas

$$a_n, \qquad \forall n \geq 1 \ , \tag{20.12}$$

are all finite. Since $a_0 = \Gamma(0)$, we can separate the divergence in the four-point function by writing

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2) \ , \tag{20.13}$$

where $\tilde{\Gamma}(p^2)$ is finite and satisfies

$$\tilde{\Gamma}(0) = 0 \ . \tag{20.14}$$

In this way we have separated the logarithmic divergence in the four-point function. In general, when Taylor expanding n-point functions the divergences will be restricted to the first few terms of the expansion. We will next apply this to the renormalization of the two- and four-point functions.

20.2 Renormalization of the Mass and Field Parameters

The renormalization procedure applied to the two-point function will result in a redefinition of the mass parameter as well as of the field itself. We start by expanding $\Sigma(p^2)$ around and arbitrary constant we call m^2. Since we start with a quantity that is quadratically divergent, one derivative will only reduce the degree of divergence, but not enough to make the integral converge. In other words we write the Taylor expansion as

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \tilde{\Sigma}(p^2) \ , \tag{20.15}$$

where $\Sigma(m^2)$ is quadratically divergent,

$$\Sigma'(m^2) = \left.\frac{\partial \Sigma(p^2)}{\partial p^2}\right|_{p^2=m^2} \ , \tag{20.16}$$

is logarithmically divergent, and $\tilde{\Sigma}(p^2)$ is finite and satisfies

$$\tilde{\Sigma}(m^2) = 0 \ , \qquad\qquad \tilde{\Sigma}'(m^2) = 0 \ . \tag{20.17}$$

If we now insert (20.15) into the summed propagator in (20.4) we have

$$\Delta_F(p) = \frac{i}{p^2 - m_0^2 - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2) - \tilde{\Sigma}(p^2) + i\epsilon} \ . \tag{20.18}$$

We want to define the renormalized mass as the pole of the propagator $\Delta_F(p)$. This translates into the renormalization condition

$$\boxed{m^2 \equiv m_0^2 + \Sigma(m^2)} \ , \tag{20.19}$$

which really acts as a definition of m^2. This gives us

$$\Delta_F(p) = \frac{i}{(p^2 - m^2)\left\{1 - \Sigma'(m^2)\right\} - \tilde{\Sigma}(p^2)} \ , \tag{20.20}$$

which is consistent with m^2 being the pole of $\Delta_F(p)$ given that from (20.17) we know that $\tilde{\Sigma}(m^2) = 0$. We see from (20.19) that since $\Sigma(m^2)$ is UV-divergent, so must be the bare mass squared m_0^2, so as to cancel the divergence and leave just the physical pole mass m^2. But the mass squared parameter is not the only one that gets renormalized. Given that both $\Sigma(m^2)$ and $\tilde{\Sigma}(p^2)$ are given to a certain order in perturbation theory defined by powers of λ_0 (first order in λ_0 in this explicit example) we can write

$$\tilde{\Sigma}(p^2) \simeq \tilde{\Sigma}(p^2)\left(1 - \Sigma'(m^2)\right) \ , \tag{20.21}$$

which means that both sides of (20.21) are equivalent to the order in λ_0 we are computing $\tilde{\Sigma}(p^2)$ and $\Sigma'(m^2)$. With this trick we can rewrite (20.20) as

$$\Delta_F(p) = \frac{i}{\left[p^2 - m^2 - \tilde{\Sigma}(p^2) + i\epsilon\right](1 - \Sigma'(m^2))}\ , \tag{20.22}$$

or, defining

$$Z_\phi \equiv \frac{1}{1 - \Sigma'(m^2)} \simeq 1 + \Sigma'(m^2) + \ldots\ , \tag{20.23}$$

we can write as

$$\Delta_F(p) = \frac{iZ_\phi}{p^2 - m^2 - \tilde{\Sigma}(p^2) + i\epsilon}\ . \tag{20.24}$$

We see that Z_ϕ gives the residue of the propagator at m^2. Since $\Sigma'(m^2)$ is UV-divergent, so it will be Z_ϕ. We can absorb this divergence by a field redefinition:

$$\phi(x) \equiv Z_\phi^{-1/2}\phi_0(x)\ , \tag{20.25}$$

where $\phi(x)$ is the renormalized field. In terms of it we can now define a renormalized propagator as given by

$$\begin{aligned} \Delta_F^r(p) &= \int d^4x\, e^{ip\cdot x}\, \langle 0|T\phi(x)\phi(0)|0\rangle \\ &= Z_\phi^{-1}\int d^4x\, e^{ip\cdot x}\, \langle 0|T\phi_0(x)\phi_0(0)|0\rangle\ , \end{aligned} \tag{20.26}$$

which results in

$$\boxed{\Delta_F^r(p) = Z_\phi^{-1}\Delta_F(p) = \frac{i}{p^2 - m^2 - \tilde{\Sigma}(p^2) + i\epsilon}}\ . \tag{20.27}$$

The renormalized propagator in (20.27) is finite and it has a pole in the renormalized mass squared m^2, with a residue equal to unity. The factor Z_ϕ is sometimes referred to as the wave-function renormalization. So we see that the divergences in the loop contributions to $-i\Sigma(p^2)$ were removed by the renormalization of the mass squared and of the field.

20.3 Coupling Constant Renormalization

Before we consider the renormalization of the four-point function let us apply the result from the previous section to a generic n-point function. Using (20.25) the renormalized n-point function can be written as

$$G_r^{(n)}(p_1,\ldots,p_n) = Z_\phi^{-n/2} G_0^{(n)}(p_1,\ldots,p_n)\ , \tag{20.28}$$

where on the right-hand side we have the unrenormalized n-point function. However, if we want to obtain the amputated 1PI n-point function, on the side of the renormalized one (the left-hand side in (20.28)) we need to remove n external propagator renormalizations, i.e. since

$$\Delta_F^r(p) = Z_\phi^{-1}\,\Delta_F(p)\ , \tag{20.29}$$

we have

$$\Gamma_r^{(n)}(p_1,\ldots,p_n) Z_\phi^{-n} = Z_\phi^{-n/2}\,\Gamma_0^{(n)}(p_1,\ldots,p_n)\ , \tag{20.30}$$

where $\Gamma_0(p_1,\ldots,p_n)$ is the unrenormalized 1PI n-point function. This results in

$$\Gamma_r^{(n)}(p_1,\ldots,p_n) = Z_\phi^{n/2}\,\Gamma_0^{(n)}(p_1,\ldots,p_n)\ . \tag{20.31}$$

Turning to the four-point function up to order λ_0^2 the relevant 1PI diagrams are the tree-level one at the beginning of this chapter, and the one-loop diagrams in Figure 20.5. We can write then

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)\ , \tag{20.32}$$

with the last three terms corresponding to Figures 20.5-(a), (b) and (c) respectively, and they are given by (20.6). Since the loop corrections make the four-point amplitude depend on kinematics, the renormalization condition that we must impose in order to define the physical coupling must be at an arbitrary kinematic point. For instance, if we choose a symmetric point s_0, t_0, u_0 such that

$$s_0 = t_0 = u_0\ , \tag{20.33}$$

then since $s + t + u = 4m^2$, we have that

$$s_0 = t_0 = u_0 = \frac{4}{3}m^2 \ . \tag{20.34}$$

At this specific point we choose to define the physical coupling constant by the renormalization condition

$$\Gamma_r^{(4)}(s_0, t_0, u_0) \equiv -i\lambda \ , \tag{20.35}$$

which is really a definition of the renormalized coupling λ. In (20.35) we have the renormalized four-point function at the chosen kinematic point defining the renormalized coupling constant. If we now go back to the unrenormalized four-point function (20.32) and we separate the divergent pieces, we can write it as

$$\begin{aligned} \Gamma_0^{(4)}(s,t,u) &= -i\lambda_0 + \Gamma(s_0) + \tilde{\Gamma}(s) + \Gamma(t_0) + \tilde{\Gamma}(t) + \Gamma(u_0) + \tilde{\Gamma}(u) \\ &= -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \ . \end{aligned} \tag{20.36}$$

Since the last three terms of (20.36) are finite we want to single out the first two so as to absorb the divergences in $\Gamma(s_0)$. For this purpose we define

$$-iZ_\lambda^{-1}\lambda_0 \equiv -i\lambda_0 + 3\Gamma(s_0) \ . \tag{20.37}$$

To express the renormalized four-point function we make use of (20.31) so that

$$\Gamma_r^{(4)}(s,t,u) = Z_\phi^2\,\Gamma_0(s,t,u) \ . \tag{20.38}$$

Then we have

$$\begin{aligned} \Gamma_r^{(4)}(s_0,t_0,u_0) &= Z_\phi^2\,\Gamma_0^{(4)}(s_0,t_0,u_0) \\ -i\lambda &= Z_\phi^2\left(-iZ_\lambda^{-1}\lambda_0\right) \ , \end{aligned} \tag{20.39}$$

where in the second line we used the renormalization condition (20.35) on the left-hand side, and (20.36) and (20.37) on the right-hand side, supplemented by the fact that

$$\tilde{\Gamma}(s_0) = \tilde{\Gamma}(t_0) = \tilde{\Gamma}(u_0) = 0 \ . \tag{20.40}$$

Comparing both sides we arrive at the expression

$$\boxed{\lambda = Z_\phi^2\, Z_\lambda^{-1}\, \lambda_0}\ , \tag{20.41}$$

for the renormalized coupling. The renormalization condition (20.35) imposes that the coupling constant λ is physical, i.e. is given by the value of the measured amplitude at some arbitrary point. This means that the divergences in (20.37) present in the $\Gamma(s_0)$'s must be canceled by λ_0. Finally, we can rewrite (20.38) as

$$\begin{aligned}\Gamma_r^{(4)}(s,t,u) &= -iZ_\phi^2 Z_\lambda^{-1}\lambda_0 + Z_\phi^2\left\{\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)\right\}\\ &= -i\lambda + Z_\phi^2\left\{\tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)\right\}\ .\end{aligned} \tag{20.42}$$

However, since in perturbation theory we always have

$$\begin{aligned}Z_\phi &\simeq 1+\mathcal{O}(\lambda_0)\\ \tilde\Gamma(p^2) &\simeq \mathcal{O}(\lambda_0^2)\ ,\end{aligned} \tag{20.43}$$

we can always express the renormalized four-point function as

$$\boxed{\Gamma_r^{(4)}(s,t,u) = -i\lambda + \tilde\Gamma(s)+\tilde\Gamma(t)+\tilde\Gamma(u)+\mathcal{O}(\lambda_0^3)}\ , \tag{20.44}$$

which is finite. We see from (20.44) that the four-point amplitude carries the information of the loop effects in the form of the s,t,u dependence acquired from them. As we can see from this procedure, the choice of the point where the renormalization condition defines the coupling λ is arbitrary. But the final expression will always be the same. For different renormalization points, we will have different values of λ, but the functional dependence will be the same.

20.4 The Renormalization Program

We have seen that the renormalized theory in terms of the parameters

$$\begin{aligned}m^2 &= m_0^2+\delta m^2\\ \phi &= Z_\phi^{-1/2}\phi_0\\ \lambda &= Z_\phi^2\, Z_\lambda^{-1}\,\lambda_0\ ,\end{aligned} \tag{20.45}$$

gives us finite two and four-point 1PI correlation functions. This also applies to *reducible* but connected diagrams as the one in Figure 20.6.

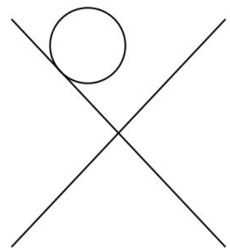


Figure 20.6: Reducible contribution to the four-point function.

This contribution can be factorized as

$$\Delta_F^r(p) \times \Gamma_r^{(4)}(s,t,u) \ , \tag{20.46}$$

which is finite. There are no new divergences in this diagram *per se*.

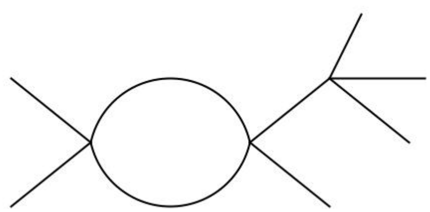


Figure 20.7: Reducible contribution to the six-point function.

The same can be said of the contribution to the six-point function depicted in Figure 20.7. However, in both cases, there are *subdivergences* associated with the infinities in the two-or the four-point functions. Since these correlation functions have been already renormalized, then all other correlation functions of the theory are now well defined and finite. In general, the result of the renormalization procedure gives us renormalized correlation functions, i.e. they depend on the renormalized parameters, in our example m^2, λ and the renormalized field. Then we can write the formal relation

$$G_r^{(n)}(p_1,\ldots,p_n,m^2,\lambda) = Z_\phi^{-n/2}\, G_0^{(n)}(p_1,\ldots,p_n,m_0^2,\lambda_0,\Lambda) \ , \tag{20.47}$$

where in general the unrenormalized correlation functions will depend not only on the unrenormalized parameters, but also on a regulator, which we generically dubbed Λ.

In what follows we will introduce a formalism that incorporates all of this in a more systematic way, renormalization by counterterms.

Additional suggested readings

- *Gauge Theory of Elementary Particle Physics*, by T.-P. Cheng and L. -F. Li, Section 2.1.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, see discussions in Sections 8.1 and 8.2.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Sections III.1 to III.3.