Lecture 2

Classical Field Theory

In this lecture we review some of the basics of classical field theory. But first we back up a little and review a few things in classical mechanics.

2.1 Stationary Action in Classical Mechanics

We consider a system of N particles with positions

$$q : \{q_1, \cdots, q_N\} \tag{2.1}$$

and velocities

$$\dot{q} : \{\dot{q}_1, \cdots, \dot{q}_N\} \tag{2.2}$$

both evaluated at some time t. The action of the system is given by

$$S = \int_{t_1}^{t_2} dt\, L[q(t), \dot{q}(t)] \ , \tag{2.3}$$

where $L[q(t), \dot{q}(t)]$ is the Lagrangian. The physical trajectory $q(t)$ between the two fixed points at t_1 and t_2 is the one that makes S stationary.

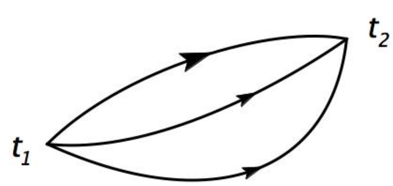

Figure 2.1: All possible trajectories contribute to S.

S is a *functional* of the trajectories $q(t)$ with boundary conditions at t_1 and t_2 that fix $q(t_1) = q_1$ and $q(t_2) = q_2$. The variation of the action is given by

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q(t)} \, \delta q(t) + \frac{\partial L}{\partial \dot{q}} \, \delta \dot{q}(t) \right\} \tag{2.4}$$

But the variation of the velocities can be written as

$$\delta \dot{q}(t) = \frac{d}{dt} \, \delta q(t) \ , \tag{2.5}$$

so we can rewrite δS as

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q(t)} \, \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \, \frac{d}{dt} \, \delta q(t) \right\} \ . \tag{2.6}$$

Then, integrating by parts we get

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \left(\frac{\partial L}{\partial q(t)} \ - \ \frac{d}{dt} \, \frac{\partial L}{\partial \dot{q}(t)} \right) \delta q(t) \ + \ \frac{d}{dt} \, \left(\frac{\partial L}{\partial \dot{q}(t)} \, \delta q(t) \right) \right\} \ , \tag{2.7}$$

or

$$\delta S = \int_{t_1}^{t_2} dt \, \left(\frac{\partial L}{\partial q(t)} \ - \ \frac{d}{dt} \, \frac{\partial L}{\partial \dot{q}(t)} \right) \delta q(t) \ + \ \left[\frac{\partial L}{\partial \dot{q}(t)} \, \delta q(t) \right]_{t_1}^{t_2} \ . \tag{2.8}$$

But the trajectories are fixed at t_1 and t_2. These are the boundary conditions we imposed. This means that $\delta q(t_1) = 0 = \delta q(t_2)$. Thus, if we impose that the action is extremal we obtain

$$\delta S = 0 \qquad \Longrightarrow \qquad \boxed{\frac{\partial L}{\partial q(t)} - \frac{d}{dt} \, \frac{\partial L}{\partial \dot{q}(t)} = 0} \ . \tag{2.9}$$

The set of equations (2.9) are referred to as the Eüler-Lagrange equations. They are the equations of motion of the system. For instance, they lead to Newton's second law if the system is a particle of mass m.

2.2 Hamiltonian Formulation

We define conjugate momenta as

$$p_i \equiv \frac{\partial L(q,\dot{q})}{\partial \dot{q}_i} \ , \tag{2.10}$$

such that now we can write the velocities as functions of p and q

$$\boxed{\dot{q} = \dot{q}(p,q)} \tag{2.11}$$

The Hamiltonian is defined through a Legendre transformation as

$$H(p,q) \ \equiv \ p_i\,\dot{q}_i - L(q,\dot{q}) \ , \tag{2.12}$$

where we are assuming Einstein summation convention in the presence of repeated indexes. Differentiating (2.12) we have

$$\begin{aligned} dH \ &= \ \dot{q}_i\,dp_i + \frac{\partial \dot{q}_j}{\partial p_i}\,p_j\,dp_i - \frac{\partial L}{\partial \dot{q}_j}\,\frac{\partial \dot{q}_j}{\partial p_i}\,dp_i \\ &\quad + \left(-\frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_j}\,\frac{\partial \dot{q}_j}{\partial q_i} + p_j\,\frac{\partial \dot{q}_j}{\partial q_i}\right) dq_i \end{aligned} \tag{2.13}$$

We can neatly rearrange (2.13) as

$$\begin{aligned} dH \ &= \ \left(\dot{q}_i + \frac{\partial \dot{q}_j}{\partial p_i}\left(p_j - \frac{\partial L}{\partial \dot{q}_j}\right)\right) dp_i \\ &+ \ \left(-\frac{\partial L}{\partial q_i} - \frac{\partial \dot{q}_j}{\partial q_i}\left(\frac{\partial L}{\partial \dot{q}_j} - p_j\right)\right) dq_i \ . \end{aligned} \tag{2.14}$$

Now using the definition of conjugate momenta (2.10) we obtain Hamilton's equations

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \ ,} \tag{2.15}$$

where we have used the Eüler-Lagrange equations (2.9) in order to obtain the second equation in (2.15).

In going to field theory, we will use the Lagrangian formulation most of the time, mostly because in relativistic quantum field theory this is manifestly covariant and the symmetries of the system are more transparently identified. However, there will be many instances when we will need to go to the Hamiltonian formulation.

2.3 Classical Fields

Here we start by considering a field or set of fields $\phi(x)$, where x is the spacetime position. The Lagrangian is a functional of $\phi(x)$ and its derivatives

$$\frac{\partial\phi(x)}{\partial x^\mu} = \partial_\mu\phi(x) \ . \tag{2.16}$$

Here $\phi(x)$ can be a set of fields with an internal index i, such that

$$\phi(x) = \{\phi_i(x)\} \ . \tag{2.17}$$

We will start with the Lagrangian formulation. We define the Lagrangian density $\mathcal{L}(\phi(x), \partial_\mu\phi(x))$ by

$$L = \int d^3x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)). \tag{2.18}$$

In this way the action is

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)) \ , \tag{2.19}$$

where we are again using the Lorentz invariant spacetime volume element

$$d^4x = dt\, d^3x \ . \tag{2.20}$$

From (2.19) is clear that $\mathcal{L}$ must be Lorentz invariant. In addition, $\mathcal{L}$ might also be invariant under other symmetries of the particular theory we are studying. These are generally called internal symmetries and we will study them in more detail later in this lecture, and a lot more in the rest of the course.

We now wish to vary the action in (2.19) in order to find the extremal solutions and obtain the equations of motion, just as we did in the case of a system of N particles. Here we get

$$\delta S = \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta(\partial_\mu\phi) \right\} \tag{2.21}$$

But, analogously to what we did in the previous section, we have that

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) \tag{2.22}$$

Then, the variation of the action is

$$\begin{aligned} \delta S &= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\delta\phi) \right\} , \\ &= \int d^4x \left\{ \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right) \delta\phi + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \right) \right\} . \end{aligned} \tag{2.23}$$

In the second line in (2.23) we have integrated by parts. The last term is a four-divergence, i.e. a total derivative. Since the integral is over the volume of all of spacetime, the resulting (hyper-)surface term must be evaluated at infinity. But just as in the case of N particle dynamics, the value of the field variation at these extremes is $\delta\phi = 0$. Thus, the (hyper-)surface term in (2.23) does not contribute.

Then imposing $\delta S = 0$, we see that the first term in (2.23) multiplying $\delta\phi$ must vanish for all possible values of $\delta\phi$. We obtain

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0} , \tag{2.24}$$

which are the Eüler-Lagrange equations, one for each of the $\phi_i(x)$, also known as equations of motion.

If now we want to go to the Hamiltonian formulation, we start by defining the canonically conjugated momentum by

$$p(x) = \frac{\partial L}{\partial\dot\phi(x)} = \frac{\partial}{\partial\dot\phi(x)} \int d^3y\,\mathcal{L}(\phi(y), \partial_\mu\phi(y)) , \tag{2.25}$$

which results in the momentum density

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi(x)} . \tag{2.26}$$

Here $\pi(x)$ is the momentum density canonically conjugated to $\phi(x)$. Then the Hamiltonian is given by

$$H = \int d^3x\,\pi(x)\,\dot\phi(x) - L , \tag{2.27}$$

which leads to the Hamiltonian density

$$\mathcal{H}(x) = \pi(x)\,\dot{\phi}(x) - \mathcal{L}(x) \;, \tag{2.28}$$

where we must remember that we evaluate at a fixed time t, i.e. $x = (t, \mathbf{x})$ for fixed t. The Lagrangian formulation allows for a Lorentz invariant treatment. On the other hand, the Hamiltonian formulation might have some advantages. For instance, it allows us to impose canonical quantization rules.

Example: We start with a simple example: the non-interacting theory of real scalar field. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,m^2\,\phi^2 \;, \tag{2.29}$$

We will call the first term in (2.29) the kinetic term. In the second term m is the mass parameter, so this we will call the mass term. We first obtain the equations of motion by using the Eüler-Lagrange equations (2.9). We have

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\,\phi \;, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\phi \;, \tag{2.30}$$

giving us

$$\boxed{(\,\partial^2 + m^2)\,\phi = 0\,} \;, \tag{2.31}$$

where the D'Alembertian operator is defined by $\partial^2 = \partial_\mu\partial^\mu$. The equation of motion (2.31) is called the Klein-Gordon equation. This might be a good point for a comment. In "deriving" the equations of motion (2.31) , we started with the "given" Lagrangian density (2.29). But in general this is not how it works. Many times we have information that leads to the equations of motion, so we can guess the Lagrangian that would correspond to them. This would be a bottom up construction of the theory. In this case, the Klein-Gordon equation is just the relativistic dispersion relation $p^2 = m^2$, noting that $-i\partial_\mu = p_\mu$. So we could have guessed (2.31), and then derive $\mathcal{L}$. However, we can invert the argument: the Lagrangian density (2.29) is the most general non-interacting Lagrangian for a real scalar field of mass m that respects Lorentz invariance. So imposing the symmetry restriction on $\mathcal{L}$ we can build it and then really derive the equations of motion. In general, this procedure of writing down the most general Lagrangian density consistent with all the symmetries of the theory will be limiting enough to get the right dynamics[1].

[1]Actually, in the presence of interactions we need to add one more restriction called renormalizability. Otherwise, in general there will be infinite terms compatible with the symmetries.

Now we want to derive the form of the Hamiltonian in this example. It is convenient to first write the Lagrangian density (2.29) as

$$\mathcal{L} = \frac{1}{2}\,\dot{\phi}^2 - \frac{1}{2}\,(\vec{\nabla}\phi)^2 - \frac{1}{2}\,m^2\phi^2 \ . \tag{2.32}$$

The canonically conjugated momentum density is now

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \ . \tag{2.33}$$

Then, using (2.28) we obtain the Hamiltonian

$$H = \int d^3x \,\left\{\pi^2 - \frac{1}{2}\,\pi^2 \, + \frac{1}{2}\,(\vec{\nabla}\phi)^2 + \frac{1}{2}\,m^2\,\phi^2\right\} \tag{2.34}$$

which results in

$$\boxed{H = \int d^3x \,\left\{\frac{1}{2}\,\pi^2 \, + \frac{1}{2}\,(\vec{\nabla}\phi)^2 + \frac{1}{2}\,m^2\,\phi^2\right\}} \ . \tag{2.35}$$

We clearly identify the first term in (2.35) as the kinetic energy, the second term as the energy associated with spatial variations of the field, and finally the third term as the energy associated with the mass.

2.4 Continuous Symmetries and Noether's Theorem

In addition to being invariant under Lorentz transformations, the Lagrangian density $\mathcal{L}$ can be a scalar under other symmetry transformations. In particular, when the symmetry transformation is continuous, we can express it as an infinitesimal variation of the field $\phi(x)$ that leaves the equations of motion invariant. Let us consider the infinitesimal transformation

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \epsilon\,\Delta\phi \ , \tag{2.36}$$

where ϵ is an infinitesimal parameter. The change induced in the Lagrangian density is

$$\mathcal{L} \longrightarrow \mathcal{L} + \epsilon\,\Delta\mathcal{L} \ , \tag{2.37}$$

where we factorized ϵ for convenience in the second term. This term can be written as

$$\begin{aligned} \epsilon\,\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\,(\epsilon\,\Delta\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\epsilon\,\Delta\phi) \\ &= \epsilon\,\Delta\phi\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right\} + \epsilon\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi\right)\;. \end{aligned} \tag{2.38}$$

The first term in (2.38) vanishes when we use the equations of motion. The last term is a total derivative so it does not affect the equations of motion when we minimize the action. We can take advantage of this fact and define

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi \tag{2.39}$$

such that its four-divergence

$$\partial_\mu\, j^\mu = 0\;, \tag{2.40}$$

up to terms that are total derivatives in the action, and therefore do not contribute if we use the equations of motion. We call this object the conserved current associated with the symmetry transformation (2.36). We will illustrate this with the following example.

Example:

We consider a complex scalar field. That is, there is a real part of $\phi(x)$ and an imaginary part, such that $\phi(x)$ and $\phi^*(x)$ are distinct. The Lagrangian density can be written as

$$\mathcal{L} = \partial_\mu\phi^*\,\partial^\mu\phi - m^2\,\phi^*\phi\;. \tag{2.41}$$

The Lagrangian density in (2.41) is invariant under the following transformations

$$\phi(x) \longrightarrow e^{i\alpha}\,\phi(x) \tag{2.42}$$
$$\phi^*(x) \longrightarrow e^{-i\alpha}\,\phi^*(x)\;, \tag{2.43}$$

where α is an arbitrary constant real parameter. If we consider the case when α is infinitesimal ($\alpha \ll 1$),

$$\phi(x) \longrightarrow \phi'(x) \simeq \phi(x) + i\alpha\phi(x) \tag{2.44}$$
$$\phi^*(x) \longrightarrow \phi^{*'}(x) \simeq \phi^*(x) - i\alpha\phi^*(x)\;, \tag{2.45}$$

which tells us that we can make the identifications

$$\begin{aligned} \epsilon\,\Delta\phi &= i\alpha\,\phi \\ \epsilon\,\Delta\phi^* &= -i\alpha\,\phi^*\ , \end{aligned} \tag{2.46}$$

with $\epsilon = \alpha$. In other words we have

$$\Delta\phi = i\phi\ , \qquad \Delta\phi^* = -i\phi^*\ . \tag{2.47}$$

Armed with all these we can now build the current j^μ associated with the symmetry transformations (2.43). In particular, since there are two independent degrees of freedom, ϕ and ϕ^*, we will have two terms in j^μ

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\Delta\phi^*\ , \tag{2.48}$$

From (2.41) we obtain

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi^*\ , \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \partial^\mu\phi \tag{2.49}$$

which results in

$$j^\mu = i\left\{(\partial^\mu\phi^*)\,\phi - (\partial^\mu\phi)\,\phi^*\right\}\ . \tag{2.50}$$

We would like to check current conservation, i.e. check that $\partial_\mu j^\mu = 0$. However, as we discussed above, this is only true up to total divergences that do not affect the equations of motion. So the strategy is to compute the four-divergence of the current and then use the equations of motion to see if the result vanishes. The equations of motion are easily obtained from the Eüler-Lagrange equations applied to $\mathcal{L}$ in (2.41). This results in

$$(\partial^2 + m^2)\phi^* = 0\ , \qquad (\partial^2 + m^2)\phi = 0\ , \tag{2.51}$$

i.e. both ϕ and ϕ^* obey the Klein-Gordon equation. Taking the four-divergence in (2.50) we obtain

$$\partial_\mu j^\mu = i\left\{(\partial^2\phi^*)\phi - (\partial^2\phi)\phi^*\right\}\ , \tag{2.52}$$

Thus, this is not zero in general. But applying the equations of motion in (2.51) we get

$$\partial_\mu j^\mu = i\left\{(-m^2\phi^*)\phi - (-m^2\phi)\phi^*\right\} = 0 \ , \tag{2.53}$$

which then verifies current conservation. We conclude that, at least at the classical level, as long as the equations of motion are valid, the current is conserved.

2.5 Spacetime Translation and the Energy-Momentum Tensor

We end this lecture by considering spacetime symmetry transformations. In particular, we are interested in continuous translations defined by

$$x^\mu \longrightarrow x^\mu + a^\mu \ , \tag{2.54}$$

where the components of the four-vector a^μ are infinitesimal constants. Let us consider a theory with a field $\phi(x)$. Under (2.54) it transforms as

$$\phi(x) \longrightarrow \phi(x+a) \simeq \phi(x) + a^\mu\partial_\mu\phi \ , \tag{2.55}$$

where we have basically Taylor-expanded in a. If (2.54) is a symmetry of the Lagrangian density, then it should be invariant up to a total derivative. Thus, we have

$$\mathcal{L} \longrightarrow \mathcal{L} + \Delta\mathcal{L} \simeq \mathcal{L} + a^\mu\,\partial_\mu\mathcal{L} \ . \tag{2.56}$$

The last term is the total derivative and it can be rewritten as

$$a^\nu\partial_\mu(\delta^\mu_\nu\,\mathcal{L}) \ . \tag{2.57}$$

Then, for each value of the index ν there is a conserved current associated with the symmetry transformations in (2.54). We define the energy-momentum tensor, for fixed ν, as

$$T^\mu_\nu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta^\mu_\nu\,\mathcal{L} \ . \tag{2.58}$$

In this expression we recognize the definition of the current in the first term, provided we identify $\partial_\nu\phi$ as the variation of ϕ under (2.54), as we can see in (2.55). The subtraction of the second term guarantees that T^μ_ν will be divergenceless for each value of ν. That is, it guarantees that

$$\partial_\mu T^\mu_\nu = 0 \ . \tag{2.59}$$

As we mentioned above, there will be one conserved current for each value of ν, so one conserved charge, defined by the general expression

$$Q = \int d^3x\, j^0 \ , \tag{2.60}$$

where in this case $j^0 = T^0_\nu$. This means we will have four conserved charges. The first of them is

$$H = \int d^3x\, T^{00} \ , \tag{2.61}$$

where we took $\nu = 0$. This is the Hamiltonian, the conserved charge associated with time translations. In other words, T^{00} is the Hamiltonian density. For $\nu = 1, 2, 3$ we get

$$p^i = \int d^3x\, T^{0i} \ , \tag{2.62}$$

which defines the components of the spatial momentum, the conserved charged associated with spatial translations.

Additional suggested readings

- *Course on Theoretical Physics, Vol. 1: Mechanics*, L. Landau and E. Lifshitz. In particular Chapters 1 and 2.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber. Chapter 1, has a good treatment of classical field theory.