Lecture 19

Renormalization

Virtual processes in quantum field theory will modify the parameters of a theory, i.e the parameters in the lagrangian. In perturbation theory these contributions are ordered by an expansion parameter, typically a coupling constant, in order to have a controlled approximation. For instance, in a theory with a real scalar with the lagrangian given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,m^2\,\phi^2 - \frac{\lambda}{4!}\phi^4 \;, \tag{19.1}$$

the two-point function to order λ admits the one-particle irreducible diagrams (1PI) shown in Figure 19.1.

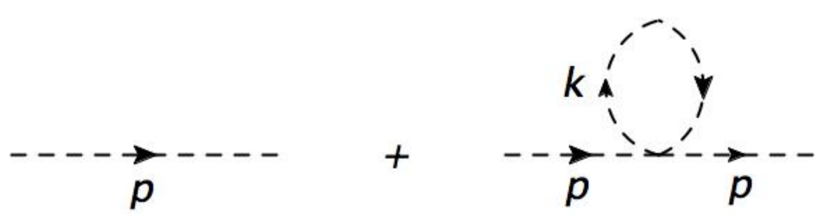

Figure 19.1: 1PI diagrams contributing to the two-point function in the theory with lagrangian (19.1), to order λ.

The first diagram is the free propagator. The second one gives a contribution to the two-point function that must be integrated over the undetermined four-momentum k, and is

$$\frac{(-i\lambda)}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\;, \tag{19.2}$$

where the factor of two is due to the symmetry of the diagram. The need for the integration is a consequence of the momentum conservation at the vertex and is consistent with the quantum mechanical character of the computation: all possible values of the four-momentum k contribute to the amplitude. The contribution form (19.2) will result in a shift of the two-point function. It will change the position of the pole of the propagator through a shift δm^2 in the parameter m^2 in (19.1), and will change the residue at the pole. The latter will be absorbed by a redefinition of the field $\phi(x)$ itself.

In addition to shifting the parameters of the theory entering in the two-point function, the one-loop diagram of Figure 19.1 diverges for large values of the momentum. This is a consequence of the fact that the momentum integration is not limited.[1] To see this clearly we rotate the time component of k_μ into Euclidean space. Then we define the Euclidean four-momentum by

$$k_0 \to ik_4 \Longrightarrow \quad k^2 = k_0^2 - \mathbf{k}^2 = -k_4^2 - \mathbf{k}^2 \equiv -k_E^2 \ , \tag{19.3}$$

such that now the integral in (19.2) can be written in terms of the 4D Euclidean momentum k_E as

$$\frac{(-i\lambda)}{2}\int \frac{d^4k_E}{(2\pi)^4}\,\frac{1}{k_E^2+m^2} = \frac{(-i\lambda)}{2}\int \frac{dk_E\,k_E^3\,d\Omega_E}{(2\pi)^4}\,\frac{1}{k_E^2+m^2}\ , \tag{19.4}$$

where the 4D solid angle is $\Omega_E = 2\pi^2$.[2] Finally, the remaining Euclidean momentum integral can be cutoff at some value Λ giving

$$\frac{(-i\lambda)}{16\pi^2}\int_0^\Lambda \frac{dk_E\,k_E^3}{k_E^2+m^2} \simeq -i\frac{\lambda}{32\pi^2}\,\Lambda^2 + \dots\ , \tag{19.5}$$

where the dots denote terms diverging with less than two powers of Λ, or terms that are finite after the limit $\Lambda \to \infty$ is taken.

Similarly, the 1PI contributions to the four-point function up to order λ^2 include loop diagrams as the ones shown in Figure 19.2.

The loop contribution shown is

[1] In some applications, such as in condensed matter systems, this is not the case. The maximum momentum (or minimum distance between two points) is a physically meaningful concept resulting in a finite value for (19.2).

[2] You may need to think a bit about this. We will derive a general expression later on.

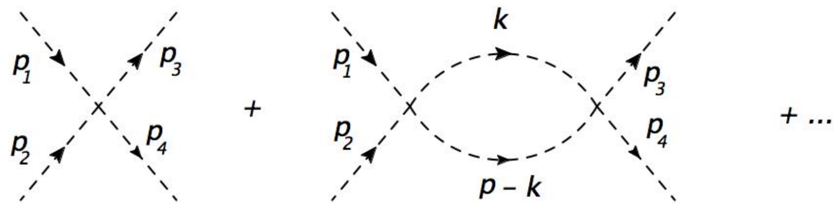

Figure 19.2: 1PI diagrams contributing to the four-point function in the theory with lagrangian (19.1), to order λ^2. Shown is only one of three order-λ^2 1PI loop diagrams. Here $p \equiv p_1 + p_2$.

$$\begin{aligned} & \frac{-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(p-k)^2 - m^2} \,, \\ = \; & \frac{i\lambda^2}{16\pi^2} \int_0^\Lambda \frac{dk_E k_E^3}{(k_E^2 + m^2)\,(p - k_E)^2 + m^2)} \,, \\ \simeq \; & \frac{i\lambda^2}{16\pi^2} \ln \Lambda^2 + \ldots \,, \end{aligned} \tag{19.6}$$

with the dots denoting terms finite in the $\Lambda \to \infty$ limit. So these one loop contributions, due to the presence of two propagators bringing two more powers of k in the denominator, are logarithmically divergent. The presence of the cutoff Λ regulates the ultra-violet (UV) behavior of the integral and gives us an idea of the "amount" of divergence we are dealing with. However, both loop diagrams are still divergent.

UV divergences like these are always present in relativistic quantum field theory. They come from the fact that undetermined momenta can be as large as possible, or the distance between any two positions in spacetime can be made as small as possible. Although their presence requires care, it is still possible to define the changes in the theory due to the quantum corrections in loop diagrams. The process of regularizing divergences is part of the renormalization procedure. Renormalization redefines all the paramenters of a theory in the presence of interactions. That is, as in our example, redefinitions of m, λ and the field $\phi(x)$ itself. Furthermore, and as we will see below, in renormalizable theories it will always be possible to absorb all divergences in a *finite* number of parameters. In order to fix how the renormalized parameters are defined and how the divergences are absorbed, we will need to impose the so-called renormalization conditions. These are external conditions, often times determined by experimental facts or convenience, that will be imposed on the parameters of the theory. Let us give an example valid for the renormalization of the scalar mass in (19.1).

$$m^2_{\text{ren.}} = \delta m^2_{\text{loop}} + \Delta m^2_{\text{ct}} \,, \tag{19.7}$$

where the renormalized scalar mass will be some input value, the physical mass. The first term on the right of (19.7) is the result of the loop calculations, which is divergent. The second term is called a counter-term and is introduced in order to cancel the divergences. We must impose a *renormalization condition* in order to fix how this procedure works. In this case, we will impose that the renormalized mass be the pole of the renormalized propagator. In this example, the presence of the counter-term in (19.7) plus the renormalization condition is all we need to renormalize the mass. As we will see in later lectures, for the case of the four-point function we will follow a similar procedure for the coupling constant λ. In that case, however, the one loop diagram will leave a measurable and finite kinematic contribution to the amplitude which is purely the result of the renormalization procedure. But before we go into the details of renormalization and regularization, we should know how to decide when a theory is renormalizable.

19.1 Divergences and Renormalizability

We will develop in this section a way to classify quantum field theories according to a criterion we call renormalizability. A theory or an interaction is renormalizable if there are only a finite number of distinct divergences, which then can be absorbed by the renormalization of a finite number of parameters in the lagrangian. We will comment on the use and validity of both renormalizable and non-renormalizable theories at the end of this section. We will also see that the way we classified interactions as renormalizable or non-renormalizable is related to our discussion of dimensional analysis in a previous lecture.

We would like to classify the divergences of a given theory. Let us imagine that we label the types of interactions in it by i, and that n_{if} is the number of fields of type f present in the interaction of type i. For instance if the interaction i in question is

$$-\frac{\lambda}{4!}\phi^4 \;, \tag{19.8}$$

then for the type of field $f=\phi$ there are four fields in the interaction, i.e. $n_{i\phi}=4$. On the other hand, if the interaction is QED, i.e. is

$$-eA_\mu\bar{\psi}\gamma^\mu\psi \;, \tag{19.9}$$

then we have two types of fields: $f=\psi$ and $f=\gamma$, fermions and photons. We have $n_{i\psi}=2$ and $n_{i\gamma}=1$.

Next, we also would like to define d_i as the number of derivatives acting on fields in the interaction of type i. This will be important in what follows since each derivative introduces a power of momentum. This is the case, for instance, for the interactions in scalar QED.

Superficial Degree of Divergence

Let us assume a given Feynman diagram contains the integral

$$\int F(p,\dots)\, d^4p \; . \tag{19.10}$$

Here $F(p,\dots)$ contains all the contributions with powers of p coming from vertices, propagators, etc. We define the superficial degree of divergence D by the UV behavior of the intergral in (19.10), such that it is given by

$$\sim \int dp\, p^{D-1} \; . \tag{19.11}$$

That is, D is defined as

- $+$ the number of positive powers of the integrated momentum
- $-$ the number of negative powers (typically coming from propagators)
- $+\,4$ (from d^4p)

When compared with (19.11) we see that this implies that when

$$\begin{cases} D = 0 & \longrightarrow \quad \text{logarithmic divergence} \\ D > 0 & \longrightarrow \quad \text{linear or higher divergence} \\ D < 0 & \longrightarrow \quad \text{convergent} \end{cases} \tag{19.12}$$

We would like to compute D for a given diagram in terms of the topology of the diagram. For this, we define

- $\mathbf{I_f}$: Number of internal lines of the field of type f.
- $\mathbf{E_f}$: Number of external lines of the field of type f.
- $\mathbf{N_i}$: Number of vertices of the interaction of type i in the diagram.

Contributions to D from Internal Lines

In the limit $p \to \infty$ the propagator of a particle of type f can be written as

$$\Delta_f(p) \sim \frac{1}{p^{2-2s_f}} \; , \tag{19.13}$$

where we defined s_f as

$$s_f = \begin{cases} 0 & \text{bosons} \\ \frac{1}{2} & \text{fermions} \end{cases} \tag{19.14}$$

so that we can write

$$\Delta_f(p) \sim p^{2s_f - 2} \ . \tag{19.15}$$

In this way, the contributions from propagators to D are given by

$$\boxed{\sum_f I_f(2s_f - 2)} \ , \tag{19.16}$$

with the sum being over all the types of fields f.

Contributions to D from interactions with Derivatives

These clearly contribute to D with

$$\boxed{\sum_i N_i d_i} \ , \tag{19.17}$$

where N_i is the number of vertices of type i and the sum is over all the types of interactions present in the diagram.

Contributions to D from the Loop Integrations

Each momentum integral contributes a factor of 4 to D. Then, the total contribution is 4×(number of independent integrals). There should be an integral for each undetermined momentum in the diagram. For instance, the loop diagram in the four-point function of Figure 19.2 has only one undetermined momentum. This, despite the fact that there are two propagators and two delta functions that could be totally determining their momenta. But this is not the case due to the fact that one of the two delta functions will end up being just the overall momentum conservation. So there is in fact in this diagram one undetermined momentum. From this discussion we can see that

$$\begin{aligned} \text{Number of independent momenta} \;\; = \;\; & \text{Number of internal lines} \\ & -\text{Number of momentum} - \text{conservation } \delta \text{ functions} \\ & +1 \text{ from the overall momentum conservation } \delta \text{ function} \end{aligned}$$

This translates into a contribution to D of

$$\boxed{4 \times \left(\sum_f I_f - \sum_i N_i + 1 \right)} \ , \tag{19.18}$$

where the factor in parentheses is the number of loop integrals and the second term in it is the number of vertices.

Assembling all the contributions from (19.16), (19.17) and (19.18) we can now write D as

$$D = 4 + \sum_f I_f (2s_f + 2) + \sum_i N_i (d_i - 4) \ . \tag{19.19}$$

However, this expression has too much dependence on the details of the diagram, such as the number of internal lines. These can be eliminated in favor of the number of external lines and the number of fields in each vertex. The relation is simply fixed by the topology of the diagram, since all lines come from or go to at least one vertex, with the internal lines connected to two vertices. Then we have that for a given type of field f

$$2\, I_f + E_f = \sum_i N_i \, n_{if} \ . \tag{19.20}$$

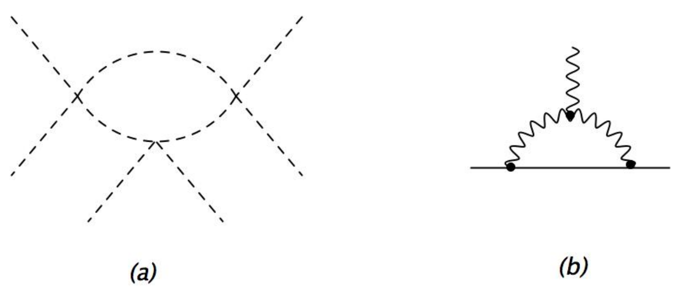

Figure 19.3: Examples of Feynman diagrams illustrating (19.20). The diagram on the left corresponds to the theory with lagrangian (19.1), to order λ^3, whereas the diagram on the right corresponds to a non-abelian gauge theory such as QCD, where the gauge bosons have self-interactions.

For instance, let us consider the diagram of Figure 19.3-(a) contributing to the six-point function in the theory of (19.1). There is only one type of field in the diagram, ϕ. The number of vertices in it is N_i=3, with $n_{if} = 4$ fields in each vertex. There are $I_f = 3$ propagators and $E_f = 6$ external lines. Replacing these values in (19.20)

$$2 \times 3 + 6 = 3 \times 4 \ , \tag{19.21}$$

we see that it is satisfied. In the case of the diagram of Figure 19.3-(b), now we have two types of fields: fermions ψ and gauge bosons g. There are also two types of vertices, with and without fermions. Then we will have two different identities defined by (19.20). For the fermions, we have $N_i = 2$ vertices containing them and $n_{if} = 2$ fermions in each of them, with only $I_\psi = 1$ fermion propagator and $E_\psi = 2$ external lines. So the identity for $f = \psi$ reads

$$2 \times 1 + 2 = 2 \times 2 \; . \tag{19.22}$$

On the other hand, for the gauge boson we have two different types of vertices: one with $n_{ig} = 1$ involving the fermions, and another one with $n_{ig} = 3$ with the self-interactions. The diagram has $E_g = 1$ external lines and $I_g = 2$ propagators of type g, so (19.20) reads

$$2 \times 2 + 1 = 3 + 2 \times 1 \; , \tag{19.23}$$

once again confirming the expression in (19.20). Making use of (19.20) we can solve for I_f as in

$$I_f = \frac{1}{2}\left(\sum_i N_i\, n_{if} - E_f\right) \; . \tag{19.24}$$

Replacing this expression for I_f in (19.19) we obtain

$$D = 4 - \sum_f E_f(s_f + 1) + \sum_f \sum_i N_i\, n_{if}\,(s_f + 1) + \sum_i N_i(d_i - 4) \; . \tag{19.25}$$

Inverting the double sum and defining

$$\Delta_i \equiv 4 - d_i - \sum_f n_{if}\,(s_f + 1) \; , \tag{19.26}$$

we obtain

$$\boxed{D = 4 - \sum_f E_f(s_f + 1) - \sum_i N_i\, \Delta_i} \; . \tag{19.27}$$

We see that the superficial degree of divergence still depends on some internal details of the diagram such as the number of vertices of the type i, N_i, summed over all the types of vertices. However, this expression is already much more useful than (19.19). To see this we notice that the quantity Δ_i defined in (19.26) is a property of the type of interaction i, so it does not depend on the diagram. Let us examine a few examples.

1. ϕ^4 Theory:

   With the interaction vertex coming from

$$-\frac{\lambda}{4!}\phi^4 \ , \tag{19.28}$$

   we will have $d_i = 0$, $s_f = 0$ and $n_{if} = 4$ so that $\Delta_i = 0$.

2. QED

   The vertex comes from

$$-eA_\mu\bar{\psi}\gamma^\mu\psi \ , \tag{19.29}$$

   so that for fermions we have $s_\psi = 1/2$, for photons $s_\gamma = 0$, $n_{i\psi} = 2$ and $n_{i\gamma} = 1$ to give

$$\Delta_i = 4 - 2(1/2 + 1) - 1(0 + 1) = 0 \ . \tag{19.30}$$

3. Four-fermion interaction

   The interaction we have in mind is

$$G\,\left(\bar{\psi}\Gamma\psi\right)\left(\bar{\psi}\Gamma\psi\right) \ , \tag{19.31}$$

   where the Γ's are generic Dirac structures, appropriately contracted with each other (e.g. $\mathbf{1}$, γ^μ, $\sigma_{\mu\nu}$, etc.). Here we have

$$\Delta_i = 4 - 4(1/2 + 1) = -2 \ . \tag{19.32}$$

At this point, you can already see that in general we can write Δ_i as

$$\Delta_i = 4 - d_{\mathcal{O}_i} \ , \tag{19.33}$$

where $d_{\mathcal{O}_i}$ is the dimension of the operator un question. For instance, for $\mathcal{O}_i = \phi^4$, or $A_\mu\bar{\psi}\gamma^\mu\psi$ we have $d_{\mathcal{O}_i} = 4$, whereas for $\mathcal{O}_i = (\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi)$ we have $d_{\mathcal{O}_i} = 6$. Thus, Δ_i corresponds to the energy dimensions of the coupling constants in each case.

It turns out that the expression (19.27) can tell us a lot about the divergence structure of a given theory once we know the Δ_i of the relevant interactions. Let us study the consequences of the different possible values of Δ_i on the renormalizability of the interactions.

$\Delta_i \geq 0$: Renormalizable Theories

From (19.27) we can see that if $\Delta_i \geq 0$ then there is an upper limit for the superficial degree of divergence

$$D \leq 4 - \sum_f E_f\,(s_f + 1)\;, \tag{19.34}$$

since the term with Δ_i contributes negatively. In this way, adding external lines makes diagrams more convergent. As a result there is a *finite* number of diagrams with divergences in theories with $\Delta_i \geq 0$. Let us consider the two examples above with $\Delta_i = 0$, the ϕ^4 theory and QED.

Let us first consider the two-point function in ϕ^4 theory. With $s_\phi = 0$ and $E_\phi = 2$. Then from (19.27) we have

$$D_2 = 4 - 2\times(0+1) = 2\;, \tag{19.35}$$

meaning that the two-point function in this theory is quadratically divergent. Notice that although we already knew this from the one-loop diagram we studied earlier, we did not use any details of the diagrams. This is a generic statement about the two-point function independently of the details of any given diagram or of the order in perturbation theory we are evaluating it.

We now move to the four-point function in this theory. We now have $E_\phi = 4$, so the superficial degree of divergence of this diagrams is

$$D_4 = 4 - 4\times(0+1) = 0\;, \tag{19.36}$$

so the four-point function is logarithmically divergent. This means that diagrams with more external legs than four will be convergent. For instance, for the six-point function we have

$$D_6 = 4 - 6\times(0+1) = -2\;. \tag{19.37}$$

We conclude that in ϕ^4 theory there are divergences only in the two-point function and the four-point function. As we will see later, these will be canceled by the renormalization of the mass and the field (in the two-point function) and the coupling λ (in the four-point function).

We now consider the situation in QED. First, the two-point function of the photon gives

$$D_{2\gamma} = 4 - 2 \times (0 + 1) = 2 \ , \tag{19.38}$$

where we used $E_\gamma = 2$ and $s_\gamma = 0$. This in principle points to a quadratic divergence. However, as we will see in detail when renormalizing QED, gauge invariance reduces this to a logarithmic divergence. In any case, this is divergent. Next we look into the fermion two-point function. We now have $s_\psi = 1/2$, which results in

$$D_{2\psi} = 4 - 2 \times (1/2 + 1) = 1 \ , \tag{19.39}$$

which means that the fermion two-point function in QED is linearly divergent. We now move to the three-point function that characterizes the interactions between two fermions and a photon. We now have $E_\gamma = 1$, $E_\psi = 2$, so

$$D_3 = 4 - 2 \times (1/2 + 1) - 1 \times (0 + 1) = 0 \ , \tag{19.40}$$

that is the three-point function is logarithmically divergent. Clearly once again, if we increase the number of external lines the resulting diagram will be finite. For instance, if we consider a diagram with two external fermions and two external photons we get

$$D_4 = 4 - 2 \times (1/2 + 1) - 2 \times (0 + 1) = -1 \ , \tag{19.41}$$

which means these diagrams are all convergent. That is, these diagrams do not introduce new divergences in addition to the ones coming from the two-point functions of photons and fermions and from the vertex.

As we see in Figure 19.4, these corrections do appear, but their divergences have already been absorbed by the renormalization of the photon and fermion fields, the fermion mass, as well as of the coupling e in the vertex. The renormalization procedure ensures that once these *finite number* of redefinitions are done, all other correlation functions in the theory will be finite and physically well defined. This is what it means for a theory to be renormalizable.

$\Delta_i < 0$: Non-Renormalizable TheoriesOn the other hand, we see that for negative values of Δ_i (19.27) shows that increasing the number of *vertices* in a diagram would increase its degree of divergence D. This means that the number of diagrams with divergences in the theory is actually infinite and we cannot absorb the divergences in a finite number

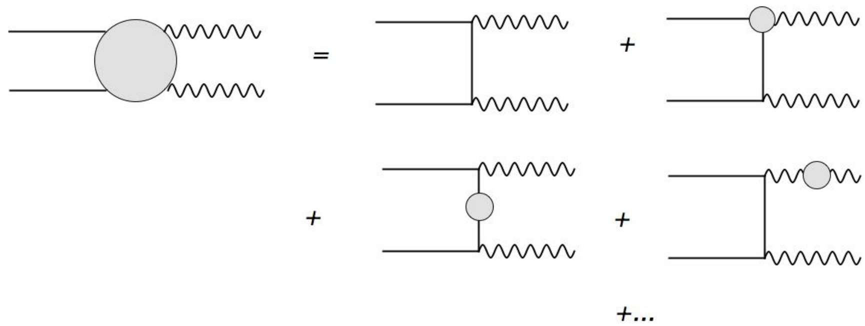

Figure 19.4: The four-point function in QED with two external fermions and two external photons. It introduces no new divergences, as seen in (19.41). The only possible divergences in it come from the ones in the two-point functions and the vertex, but these were already absorbed by renormalization.

of parameters in the lagrangian. These theories are said to be non-renormalizable. This does not mean that they are not useful. Let us consider as an example a four-fermion interaction of the form

$$\begin{aligned}\mathcal{L}_{4f} &= G\,(\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi) \\ &= \frac{g}{M^2}\,(\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi)\;,\end{aligned} \tag{19.42}$$

where $\Gamma = \mathbf{1}, \gamma_\mu, \ldots$ is some Dirac structure. In he second line of (19.42) we used the fact that G has inverse units of energy squared, to write it in terms of a dimensionless coupling g and an arbitrary mass scale M. We are interested in the four-point function necessary for fermion–antifermion scattering. At tree-level the squared of the amplitude is

$$|\mathcal{A}|^2 \sim g^2 \frac{s^2}{M^4}\;, \tag{19.43}$$

with $s = (p_1 + p_2)^2$ and we are neglecting the fermion masses. One loop contributions such as those in Figure 19.5 will renormalize the coupling g, apparently in the usual way. There are divergences coming from these diagrams that will be cancelled by making use of a renormalization condition for the amplitude.

So it looks like we can renormalize this amplitude. And in principle this is the case. The problem comes when we consider the six-point function. Since $\Delta_i = -2$ we know this is also divergent in this theory. The relevant one-loop diagrams are like the one in Figure 19.6. This requires the presence of a counter-term, i.e. we need to add to the lagrangian the term

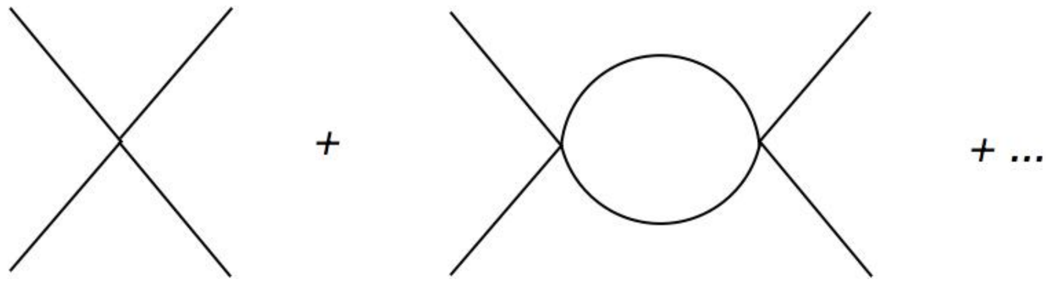

Figure 19.5: Contributions to the four-point function up to one loop in the four-fermion theory in (19.42). The dots refer to the the t and u channel one-loop diagrams similar to the one shown.

$$\frac{\gamma}{M^5}\,(\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi)\ , \tag{19.44}$$

where γ is a dimensionless constant and we are always assuming that the Γ's appropriately contract all Lorentz indices. This dimension-nine operator is necessary in order to renormalize the four-fermion dimension-six operator, since it will generate the appropriate counter-term for the diagram in Figure 19.6.

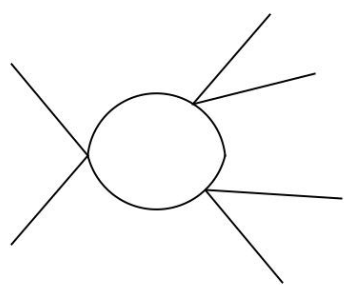

Figure 19.6: An example of a one-loop contribution to the six-point function.

Now here comes the problem: this dimension-nine operator also generates one-loop contributions to the four-point function. One example of these is shown in Figure 19.7.

These contributions now depend on the new parameter γ. So in order to renormalize the four-point function in the four-fermion theory we need to impose a new renormalization condition to fix the value of γ. It is clear that we will also get a contribution from a dimension-twelve operator that *must be present* in order to provide a counter-term for the divergent one loop eight-point function. So there will be yet a new contribution with a new undetermined coefficient that must be fixed by a new renormalization condition. This process can go on forever. The point is that in order to truly renormalize the four-fermion

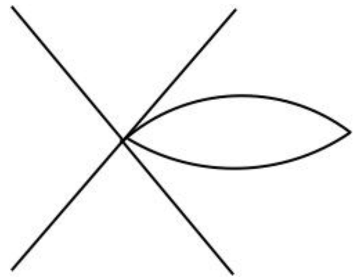

Figure 19.7: An example of a one-loop contribution to the four-point function coming from the dimension-nine operator in (19.44).

theory we need infinitely many higher dimensional operators, each of them with a new undetermined constant requiring a new renormalization condition. This is why we call $\Delta_i < 0$ theories non-renormalizable.

On the other hand, the contributions of these higher-dimensional operators are suppressed by an energy scale M. If this is much larger than the typical scale of the process being studied, then we will not need the input of their coefficients and just using a few operators with the lowest dimensions (e.g. just using (19.42)) would suffice. For instance, if we were to compute the effect of the dimension-nine operator (19.44) in the fermion-antifermion scattering process, by dimensional analysis this should be like

$$|\mathcal{A}|^2_{\text{dim}-9} \sim \gamma^2 \frac{s^5}{M^{10}}\, F(s)\ , \tag{19.45}$$

with $F(s)$ a dimensionless function of the center-of-mass energy. Contributions from higher-dimensional operators will be suppressed by higher powers of s/M^2. So as long as $s \ll M^2$, we would only need the first few operators, for example just (19.42), to describe the scattering process. Then we conclude that non-renormalizable theories can be predictive as long as the energy scale being probed by processes described by them is small compared with the scale M suppressing the infinite series of higher-dimensional operators.

There are many examples of non-renormalizable theories. Just to name a few, Chiral Perturbation Theory describes the interactions of pseudo Nambu-Goldstone Bosons (pNGB) at energies below the ρ mass. Fermi's theory of the weak interactions describes them at energies well below M_W, which mediates the exchange resulting in four-fermion interactions leading to beta decay, as well as to the decays of muons and pions. Finally, it is possible to quantize gravity and write a non-renormalizable theory involving linear fluctuations of the metric (the graviton). This quantum field theory of gravity is valid as long as the energies involved are well below the Planck scale $M_p \sim 10^{19}$ GeV.

Additional suggested readings

- *The Quantum Theory of Fields, Vol. I*, by S. Weinberg, Section 8.1 to 8.3.
- *Dynamics of the Standard Model*, J. F. Donoghue, E. Golowich and B. Holstein,