Lecture 18

Quantum Electrodynamics

Quantum Electrodynamics (QED) describes a $U(1)$ gauge theory coupled to a fermion, e.g. the electron, muon or any other charged particle. As we saw previously the lagrangian for such theory is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{18.1}$$

where the covariant derivative is defined as

$$D_\mu\psi(x) = (\partial_\mu + ieA_\mu(x))\psi(x) \ , \tag{18.2}$$

resulting in

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi \ , \tag{18.3}$$

where the last term is the interaction lagrangian and thus e is the photon-fermion coupling. We would like to derive the Feynman rules for this theory, and then compute the amplitudes and cross sections for some interesting processes.

18.1 Feynman Rules for QED

In addition to the rules for the photon and fermion propagator, we now need to derive the Feynman rule for the photon-fermion vertex. One way of doing this is to use the generating functional in presence of linearly coupled external sources

$$Z[\eta, \bar{\eta}, J_\mu] = N \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu\, e^{iS[\psi,\bar{\psi},A_\mu]+i\int d^4x\left\{J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\right\}} \ . \tag{18.4}$$

From it we can derive a three-point function with an external photon, a fermion and an antifermion. This is given by

$$G^{(3)}(x_1,x_2,x_3) = \frac{(-i)^3}{Z[0,0,0]}\;\frac{\delta^3 Z[\eta,\bar{\eta},J_\mu]}{\delta J_\mu(x_1)\delta\eta(x_2)\delta\bar{\eta}(x_3)}\bigg|_{J_\mu,\eta,\bar{\eta}=0}\;, \tag{18.5}$$

which, when we expand in the interaction term, results in

$$\begin{aligned} G^{(3)}(x_1,x_2,x_3) &= \frac{1}{Z[0,0,0]}\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu\, e^{iS_0[\psi,\bar{\psi},A_\mu]}A_\mu(x_1)\bar{\psi}(x_2)\psi(x_3) \\ &\quad \times\left(1 - ie\int d^4y A_\nu(y)\bar{\psi}(y)\gamma^\nu\psi(y) + \ldots\right) \\ &= (+ie)\int d^4y D_{F\mu\nu}(x_1-y)\mathrm{Tr}\left[\mathrm{S_F(x_3-y)\gamma^\nu S_F(x_2-y)}\right]\;, \end{aligned} \tag{18.6}$$

where S_0 is the free action, obtained from the first two terms in (18.3). To obtain the second equality we used Wick's theorem while carefully keeping the spinor indices. Finally, to obtain the vertex rule from (18.6) we assume that we are computing an amplitude for a process involving a photon, a fermion and an antifermion. Using the LSZ formalism [1] on the three-point function in (18.6) we obtain the amplitude in momentum space as

$$\langle\bar{f}(p_2)f(p_3)|A(p_1)\rangle = (-ie)(2\pi)^4\delta^{(4)}(P_1-P_2-P_3)\,\bar{u}(p_3)\gamma_\mu v(p_2)\epsilon^{\mu *}(p_1)\;, \tag{18.7}$$

where the spinor, antispinor and photon polarization appear as a consequence of the application of the LSZ formula. Since this always implies the same Feynman rules for external spinors, antispinors and polarizations, we conclude that the vertex itself will add a factor of $(-ie)\gamma^\mu$, where the Lorentz index in the gamma matrix should be contracted with that of the external photon polarization. This is illustrated in Figure 18.1.

The complete Feynman rules for QED are then obtained by having

- A factor of $\epsilon^{\mu *}(k)$ for each incoming photon of momentum k.
- A factor of $\epsilon^{\mu}(k)$ for each outgoing photon of momentum k.
- A factor of
$$\frac{i}{\not{p} - m}\;, \tag{18.8}$$
for each internal fermion line.

[1]Just as we did for the scalar-fermion theory in Lecture 14, and being just as careful with the signs coming from the i's and the action of the Klein-Gordon and Dirac differential operators on fields.

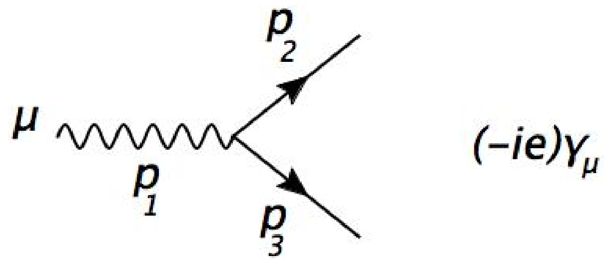

Figure 18.1: Feynman rule for the QED vertex.

- A factor of
$$\frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \,, \tag{18.9}$$
for each internal photon line (in the Feynman gauge).

Finally, the rules regarding spinors and antispinors for incoming and outgoing fermions and antifermions are the same as for those in Yukawa theory. We are now in a position to compute QED amplitudes. We will study a couple of simple examples in what follows.

18.2 $e^+e^- \to \mu^+\mu^-$ in QED

We start with a simple process in QED: electron–positron annihilation into muon pairs. As we will see, its simplicity stems from the fact that there is only one diagram contributing. The relevant lagrangian is

$$\mathcal{L} = \bar{e}(i\not{D} - m_e)e + \bar{\mu}(i\not{D} - m_\mu)\mu - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \,, \tag{18.10}$$

where it is understood that both covariant derivatives are one and the same

$$D_\alpha\psi = (\partial_\alpha + ieA_\alpha)\psi \,, \tag{18.11}$$

for $\psi = e, \mu$. Here $e(x)$ and $\mu(x)$ are the electron and muon fields, respectively. The lagrangian (18.10) implies that the gauge interactions are universal, i.e. the electron and muon interact with the photon with the same coupling[2] e. Another consequence of the form of (18.10) is that electron and muon numbers are separately conserved. This means that there are two separate *global* $U(1)$'s, one for electron and one for muons, which forbids terms such as

[2]Do no confuse with the electron field $e(x)$.

$$\bar{e} i \not{D} \mu \ , \tag{18.12}$$

violating the conservation of these charges. Taking this into account we conclude that the only Feynman diagram contributing to $e^-e^+ \to \mu^-\mu^+$ is the one shown in Figure 18.2.

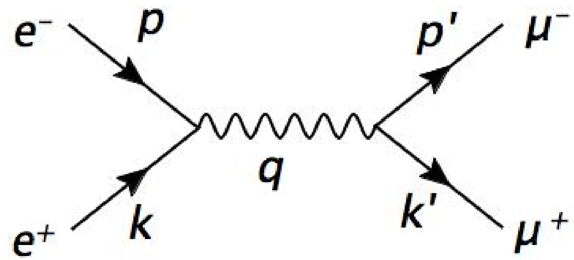

Figure 18.2: Feynman diagram for $e^-e^+ \to \mu^-\mu^+$ in QED.

Using the Feynman rules for QED from the previous section we can now write the amplitude for this process. It is

$$i\mathcal{A} = (-ie)^2 \, \bar{u}_a^{s'}(p')\gamma^\mu_{ab} v_b^{r'}(k') \, \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \, \bar{v}_c^r(k)\gamma^\nu_{cd} u_d^s(p) \ , \tag{18.13}$$

where we have shown explicitly all spinor indices and we assume the repeated ones are summed over, and we are still using the Feynman gauge for the photon propagator. To compute the cross section we will need the square of the amplitude in (18.13). For this we need to be careful about complex conjugation of the whole amplitude. We will need to compute the adjoint of the two terms to the sides of the photon propagator separately, since their spinor indices are separately contracted. That is, we need

$$(\bar{u}\gamma^\mu v)^\dagger = \left(u^\dagger \gamma^0 \gamma^\mu v\right)^\dagger = v^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger u \ . \tag{18.14}$$

But remembering that $(\gamma^0)^\dagger = \gamma^0$, and noticing that

$$(\gamma)^\dagger = \begin{cases} \gamma^\mu & \text{if} \quad \mu = 0 \\ -\gamma^\mu & \text{if} \quad \mu = 1,2,3 \end{cases} \ , \tag{18.15}$$

we obtain that

$$(\bar{u}\gamma^\mu v)^\dagger = v^\dagger \gamma^0 \gamma^\mu u = \bar{v}\gamma^\mu u \ . \tag{18.16}$$

Similarly,

$$(\bar{v}\gamma^\mu u)^\dagger = (\bar{u}\gamma^\mu v) \ . \tag{18.17}$$

Then, the complex conjugate of the amplitude (18.13) is

$$-i\mathcal{A}^\dagger = (-ie)^2\, \bar{v}_e^{r'}(k')\gamma^\alpha_{ef} u_f^{s'}(p')\, \frac{ig_{\alpha\beta}}{q^2+i\epsilon}\, \bar{u}_g^s(p)\gamma^\beta_{gh} v_h^r(k) \ . \tag{18.18}$$

We can compute the amplitude squared by keeping all the spinor indices explicitly. We obtain

$$|\mathcal{A}|^2 = \frac{e^4}{q^4}\, u_f^{s'}(p')\bar{u}_a^{s'}(p')\ v_b^{r'}(k')\bar{v}_e^{r'}(k')\, \gamma^\mu_{ab}\, \gamma^\alpha_{ef}\ u_d^s(p)\bar{u}_g^s(p)\ v_h^r(k)\bar{v}_c^r(k)\ (\gamma_\mu)_{cd}\, (\gamma_\alpha)_{gh} \ . \tag{18.19}$$

If we now assume that all spin indices $(s, r, s'r')$ are summed over, both for the initial and for the final states, we can use the relations

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \ , \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \ . \tag{18.20}$$

Using them in (18.19) we obtain

$$|\mathcal{A}|^2 = \frac{e^4}{q^4}\, (\not{p}' + m_\mu)_{fa}\, \gamma^\mu_{ab}\, (\not{k}' - m_\mu)_{be}\, \gamma^\alpha_{ef}\, (\not{p} + m_e)_{dg}\, (\gamma_\alpha)_{gh}\, (\not{k} - m_e)_{hc}\, (\gamma_\mu)_{cd} \ , \tag{18.21}$$

where all the spinor indices are summed. We notice that we can write the amplitude squared as the product of two traces over these indices as in

$$|\mathcal{A}|^2 = \frac{e^4}{q^4}\, \mathrm{Tr}\left[(\not{p}' + \mathrm{m}_\mu)\gamma^\mu(\not{k}' - \mathrm{m}_\mu)\gamma^\alpha\right] \times \mathrm{Tr}\left[(\not{p} + \mathrm{m}_\mathrm{e})\gamma_\alpha(\not{k} - \mathrm{m}_\mathrm{e})\gamma_\mu\right] \ , \tag{18.22}$$

In order to continue we need to compute the traces of the product of up to four gamma matrices. Let us start with the second factor in (18.22)

$$
\begin{aligned}
\mathrm{Tr}\left[(\not{p}+\mathrm{m_e})\gamma_\alpha(\not{k}-\mathrm{m_e})\gamma_\mu\right] &= \mathrm{Tr}\left[\not{p}\gamma_\alpha\not{k}\gamma_\mu\right] - \mathrm{m_e^2}\mathrm{Tr}\left[\gamma_\alpha\gamma_\mu\right] \\
&= p^\sigma k^\rho \mathrm{Tr}\left[\gamma_\sigma\gamma_\alpha\gamma_\rho\gamma_\mu\right] - 4\mathrm{m_e^2 g_{\mu\alpha}} \\
&= 4p^\sigma k^\rho \left\{g_{\sigma\alpha}g_{\rho\mu} - g_{\mu\alpha}g_{\rho\sigma} + g_{\mu\sigma}g_{\rho\alpha}\right\} - 4m_e^2 g_{\mu\alpha} \\
&= 4\left\{p_\alpha k_\mu - g_{\mu\alpha}p\cdot k + p_\mu k_\alpha - m_e^2 g_{\mu\alpha}\right\} \ .
\end{aligned}
\tag{18.23}
$$

In the first equality in (18.23) we use the fact that the trace over an odd number of gamma matrices vanishes. The rest of the computations simply makes repeated use of the Clifford algebra of gamma matrices

$$
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \tag{18.24}
$$

from which we can derive (exercise)

$$
\mathrm{Tr}\left[\gamma_\sigma\gamma_\alpha\gamma_\rho\gamma_\mu\right] = 4\ \left\{\mathrm{g_{\sigma\alpha}g_{\rho\mu} - g_{\sigma\rho}g_{\alpha\mu} + g_{\sigma\mu}g_{\alpha\rho}}\right\} \ . \tag{18.25}
$$

It is straightforward to obtain a similar result for the first factor in (18.22). The amplitude squared is

$$
|\mathcal{A}|^2 = 16\frac{e^4}{q^4}\ \left\{p'^\alpha k'^\mu - g^{\mu\alpha}p'\cdot k' + p'^\mu k'^\alpha - m_\mu^2 g^{\mu\alpha}\right\}\left\{p_\alpha k_\mu - g_{\mu\alpha}p\cdot k + p_\mu k_\alpha - m_e^2 g_{\mu\alpha}\right\} \ . \tag{18.26}
$$

In order to simplify the result we take the limit $m_e = 0$. This is typically appropriate since $m_\mu \gg m_e$, and all energies and momenta must be at least of the order of the muon mass for the process to occur. In this limit we have

$$
|\mathcal{A}|^2 = 32\frac{e^4}{q^4}\left\{(p\cdot p')(k\cdot k') + (p'\cdot k)(p\cdot k') + m_\mu^2 p\cdot k\right\} \ . \tag{18.27}
$$

Now that we have the amplitude squared in terms of Lorentz-invariant products of the four-momenta, we can either go to a particular frame or express these products in terms of Mandelstam variables.

<u>Center of Momentum Frame</u>:

Assuming $m_e = 0$, the four-momenta can be written in the CM frame as

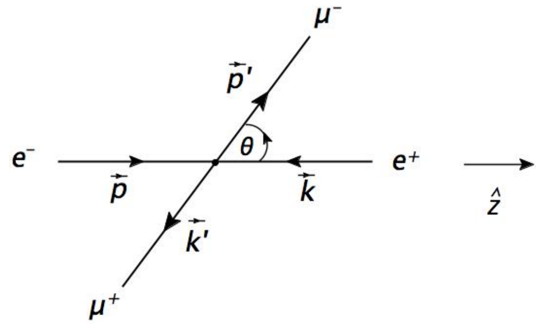

Figure 18.3: $e^-e^+ \to \mu^-\mu^+$ in the CM frame.

$$p = (E, E\hat{z}), \qquad k = (E, -E\hat{z})$$
$$p' = (E', \mathbf{p}'), \qquad k' = (E', -\mathbf{p}') \ .$$

Given that

$$E' = E = \frac{E_{\rm CM}}{2} \ , \tag{18.28}$$

with $E_{\rm CM}$ the total CM energy, the Lorentz-invariant products entering in the amplitude squared (18.27) can be written as

$$\begin{aligned} p \cdot p' &= E^2 - E|\mathbf{p}'|\cos\theta = k \cdot k' \\ p \cdot k' &= E^2 + E|\mathbf{p}'|\cos\theta = p' \cdot k \ , \end{aligned} \tag{18.29}$$

where $|\mathbf{p}'| = \sqrt{E^2 - m_\mu^2}$. Using these results in (18.27) we obtain

$$\overline{|\mathcal{A}|^2} = e^4 \left(1 + \cos^2\theta + \frac{4m_\mu^2}{E_{\rm CM}^2}\sin^2\theta \right) \ , \tag{18.30}$$

where we averaged over the initial state spins so that

$$\overline{|\mathcal{A}|^2} = \frac{1}{4}|\mathcal{A}|^2 \ . \tag{18.31}$$

We see from (18.30) that the amplitude squared has a non-trivial angular dependence, unlike the analogous process studied in Yukawa theory. To compute the differential cross section we use

$$\frac{d\sigma}{d\Omega} = \frac{1}{E_{\rm CM}^2} \frac{1}{2} \frac{|\mathbf{p}'|}{16\pi^2 E_{\rm CM}} \overline{|\mathcal{A}|^2} \ , \tag{18.32}$$

where the factor of $1/2$ comes from having the relative velocity $|1-(-1)|$, valid in the $m_e = 0$ approximation. Since, as usually, there is no azimuthal angle dependence we can write $d\Omega = 2\pi\, d\cos\theta$, resulting in the angular distribution

$$\frac{d\sigma}{d\cos\theta} = \frac{e^4}{32\pi E_{\rm CM}^2} \sqrt{1 - \frac{4m_\mu^2}{E_{\rm CM}^2}} \left(1 + \cos^2\theta + \frac{4m_\mu^2}{E_{\rm CM}^2} \sin^2\theta\right) \ , \tag{18.33}$$

A few comments about this expression are in order, particularly in comparison with the analogous expression obtained earlier for a similar process in Yukawa theory (the annihilation of a very light fermion–antifermion pair into a heavier fermion–antifermion pair). First, we see that the threshold factor in (18.33)

$$\beta = \sqrt{1 - \frac{4m_\mu^2}{E_{\rm CM}^2}} \ , \tag{18.34}$$

is only one power of β, whereas for the process mediated by the real scalar the threshold factor was β^3. This means that the scalar-mediated annihilation is considerably more suppressed near threshold than in the QED case. One power of β is always present in the factor of the final-state momentum in the cross section kinematics. But in addition, the scalar exchange amplitude squared brings two additional powers of β. Then it is possible to differentiate scalar from photon exchange just from the threshold behavior of the cross section. Second, we can see that the angular behavior is very different in both cases.

The scalar exchange has no θ dependence, whereas there is a distinct one for QED. If for simplicity we assume that $E_{\rm CM} \gg m_\mu$ then the angular distribution behaves like

$$\frac{d\sigma}{d\cos\theta} \sim \left(1 + \cos^2\theta\right) \ , \tag{18.35}$$

which has a minimun at $\theta = \pi/2$, as schematically shown in Figure 18.4. Finally, we noticed that both (18.33) and (18.35) are symmetric with respect to $\pi/2$. This means that if we divide space in $\theta \leq \pi/2$ (forward) and $\theta > \pi/2$ (backward) and count the number of μ^- in the forward and backward regions it should be the same. This is an experimental test that shows that QED respects parity.[3]

[3] This is an important experimental distinction since the exchange of the neutral gauge bosons associated with weak interactions (Z^0) do not respect parity invariance.

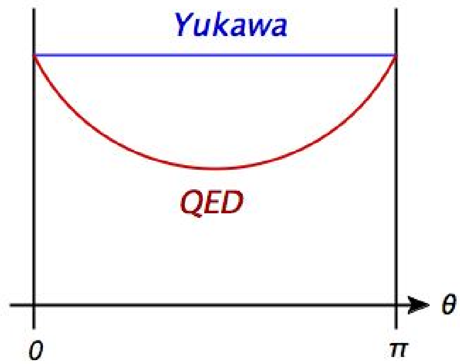

Figure 18.4: Angular distribution for $e^-e^+ \to \mu^-\mu^+$ in the CM frame, for both QED and Yukawa theory.

Mandelstam Variables: Another way to express the amplitude squared (18.27) and therefore the cross section, is by using Mandelstam variables. For the process at hand we have

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2 \\ t &= (p'-p)^2 = (k-k')^2 \\ u &= (p'-k)^2 = (k'-p)^2 \ , \end{aligned} \tag{18.36}$$

where we also identify $s = q^2$. Using $m_e = 0$, the four-vector products in (18.27) can be written as

$$\begin{aligned} p \cdot p' &= \frac{m_\mu^2 - t}{2} = k \cdot k' \\ p' \cdot k &= \frac{m_\mu^2 - u}{2} = k' \cdot p \\ p \cdot k &= \frac{s}{2} \ . \end{aligned} \tag{18.37}$$

Replacing these in (18.27) we obtain

$$\overline{|\mathcal{A}|^2} = \frac{2e^4}{s^2} \left\{ \left(m_\mu^2 - t\right)^2 + \left(m_\mu^2 - u\right)^2 - 2m_\mu^2 s \right\} \ , \tag{18.38}$$

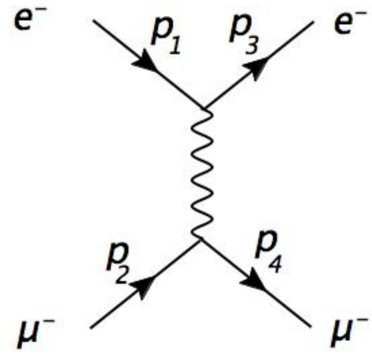

Figure 18.5: Feynman diagram for $e^-\mu^- \to e^-\mu^+$ in QED. This diagram is related by crossing symmetry with that of $e^-e^+ \to \mu^-\mu^+$ of Figure 18.2.

where we have already divided by the equally probable number of initial states. Thus, the Lorentz-invariant amplitude can be fully expressed in terms of the Mandelstam variables.

18.3 Crossing Symmetry and $e^-\mu^- \to e^-\mu^-$

We finally consider the related process of $e^-\mu^-$ scattering. From a simple inspection of the only Feynman diagram contributing as drawn in Figure 18.5 we can see that the amplitude for this process will be related to the one for $e^-e^+ \to \mu^-\mu^+$. We can obtain the amplitude squared for this process following the same steps as for the electron positron annihilation process before. The result is

$$|\mathcal{A}|^2 = \frac{e^4}{q^4}\,\mathrm{Tr}\left[(\not{p}_4 + \mathrm{m}_\mu)\gamma^\mu(\not{p}_2 + \mathrm{m}_\mu)\gamma^\alpha\right] \times \mathrm{Tr}\left[(\not{p}_3 + \mathrm{m_e})\gamma_\alpha(\not{p}_1 + \mathrm{m_e})\gamma_\mu\right] \;. \tag{18.39}$$

If we compared this expression with (18.22) obtained for $e^-e^+ \to \mu^-\mu^+$ we see that (18.39) can be obtained from that one by replacing the four-momenta as

$$\begin{aligned} p &\to p_1 & p' &\to p_4 \\ k &\to -p_3 & k' &\to -p_2 \end{aligned} \tag{18.40}$$

with the overall minus signs of each trace canceling. The replacement in (18.40) is equivalent to turning antifermions into fermions and changing the signs of their momenta, leaving the signs of the momenta of fermions that did not change. It is always possible to do this to obtain relations between the squares of amplitudes of processes obtained by these changes. This is called crossing symmetry. For this particular case, it means that the Mandelstam variables defined for the annihilation process will now change for the scattering process as

$$\begin{array}{lcl} s = (p+k)^2 & \longrightarrow & (p_3 - p_1)^2 = t \\ t = (p' - p)^2 & \longrightarrow & (p_4 - p_1)^2 = u \\ u = (p' - k)^2 & \longrightarrow & (p_4 + p_3)^2 = s \end{array} \tag{18.41}$$

Then, making these replacements in the annihilation expression for the amplitude squared in terms of the Mandelstam variables (18.38) we obtain

$$\overline{|\mathcal{A}|^2} = \frac{2e^4}{t^2}\left\{\left(m_\mu^2 - u\right)^2 + \left(m_\mu^2 - s\right)^2 - 2m_\mu^2 t\right\} \ , \tag{18.42}$$

which is the initial-state averaged amplitude squared for $e^-\mu^- \to e^-\mu^-$. In general, it is always possible to obtain the amplitude squared of a process from the expression of the amplitude squared for another process related by crossing symmetry. Crossing symmetry is connected to deep concepts in quantum field theory, such as the relation between unitarity, causality and the analyticity of correlation functions.

18.4 The Ward Identity and Processes with External Photons

Up to now, we have only studied QED processes where photons were intermediate states. To complete our treatment of QED we need to consider processes with external photons. Since these photons have polarizations, we will need to know what to do with polarization sums when computing the squares of amplitudes, very much in the same way we summed over initial and final state spins when computing scattering among fermions and/or antifermions.

Let us consider a process in QED with at least one external photon, with four-momentum k_μ. In general we can write the momentum-space amplitude as

$$i\mathcal{A}(k) \equiv i\epsilon_\mu^\lambda(k)\,\mathcal{M}^\mu(k) \ , \tag{18.43}$$

where $\mathcal{A}^\mu(k)$ is the rest of the amplitude after factoring out the polarization of the photon with four-momentum k_μ.

The square of the amplitude (18.43) is

$$|\mathcal{A}(k)|^2 = \sum_\lambda \epsilon_\mu^\lambda(k)\epsilon_\nu^{\lambda *}(k)\mathcal{M}^\mu(k)\mathcal{M}^\nu(k) \ , \tag{18.44}$$

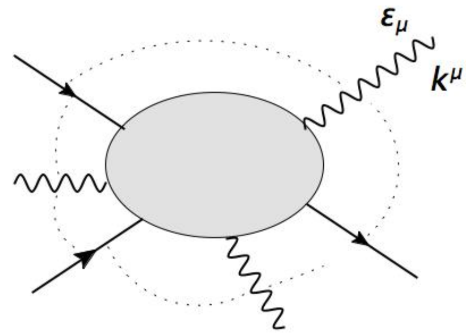

Figure 18.6: Process with an external photon. Dotted lines denote possible additional external particles.

where we are summing over all the external photon polarizations λ. But since this is an external photon it only has two polarizations which must be transverse. For instance, assuming the photon momentum to be in the $\hat{3}$ direction, we can write

$$\epsilon^1_\mu = (0, 1, 0, 0), \qquad \epsilon^2_\mu = (0, 0, 1, 0) \ . \tag{18.45}$$

This gives us for (18.44)

$$|\mathcal{A}(k)|^2 = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 \ , \tag{18.46}$$

where now the superscripts on the right-hand side refer to the Lorentz indices, i.e. directions $\hat{1}$ and $\hat{2}$ in Minkowski.

On the other hand, we know that in QED the photon couples to a conserved current as in

$$\int d^4x A_\mu(x) j^\mu(x) \ , \tag{18.47}$$

where $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. This means that our external photon in Figure 18.6 is coupled to the matrix element

$$\langle A|j^\mu(x)|B\rangle = \int d^4k e^{-ik\cdot x}\mathcal{M}^\mu(k) \ , \tag{18.48}$$

where the states $|A\rangle$ and $|B\rangle$ include all external particles in the diagram except the photon of momentum k_μ. In other words, the amplitude $\mathcal{M}^\mu(k)$ is the Fourier transform of this matrix element of the conserved current. Since $j^\mu(x)$ satisfies current conservation, i.e.

$$\partial_\mu j^\mu(x) = 0 \ , \tag{18.49}$$

we can apply it to (18.48) to obtain

$$\langle A|\partial_\mu j^\mu(x)|B\rangle = \int d^4k e^{-ik\cdot x}(-i)k_\mu \mathcal{M}^\mu(k) = 0 \ , \tag{18.50}$$

implying that

$$\boxed{k_\mu \mathcal{M}^\mu(k) = 0} \ . \tag{18.51}$$

This is known as the Ward identity. It states that replacing the polarization of an external photon by its four-momentum in an amplitude such as (18.43) must give zero. Is a result that reflects current conservation an therefore gauge invariance.

If we now go back to (18.46), and assume as before that the photon momentum is $\mathbf{k} = |\mathbf{k}|\,\hat{3}$, we have

$$k_\mu = (|\mathbf{k}|, 0, 0, |\mathbf{k}|) \ , \tag{18.52}$$

such that it satisfies $k^2 = 0$. Then the Ward identity (18.51) implies

$$k_\mu \mathcal{M}^\mu(k) = |\mathbf{k}|\,\left[\mathcal{M}^0 - \mathcal{M}^3\right] = 0 \ , \tag{18.53}$$

resulting in

$$\mathcal{M}^0 = \mathcal{M}^3 \ . \tag{18.54}$$

Here, the superscripts indicate the time and longitudinal directions, respectively. This means that now we can write the amplitude squared (18.46) as

$$|\mathcal{A}(k)|^2 = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 + |\mathcal{M}^3(k)|^2 - |\mathcal{M}^0(k)|^2 \ , \tag{18.55}$$

given that by virtue of (18.54) the last two terms add up to zero. Clearly, the right-hand side of (18.55) can be written in terms of the metric $g_{\mu\nu}$:

$$|\mathcal{A}(k)|^2 = -g_{\mu\nu}\,\mathcal{M}^\mu(k)\,\mathcal{M}^\nu\ . \tag{18.56}$$

Comparing with (18.44) we see that we can make the replacement

$$\boxed{\sum_\lambda \epsilon^\lambda_\mu(k)\epsilon^{\lambda *}_\nu(k) \longrightarrow -g_{\mu\nu}}\ , \tag{18.57}$$

when computing the amplitude squared. This replacement can always be made by virtue of the Ward identity (18.51). Although it appears that in (18.57) we are including polarizations that have time and longitudinal components, i.e. unphysical polarizations for an external photon, the Ward identity ensures that they do not contribute to the actual amplitude squared. The polarization sums (18.57) are particularly useful in computing cross sections for processes with external photons, as we will see in the following section.

18.5 Compton Scattering

We consider the scattering of a photon off an electron, $\gamma e^- \to \gamma e^-$. The two contribution diagrams are depicted in Figure 18.7.

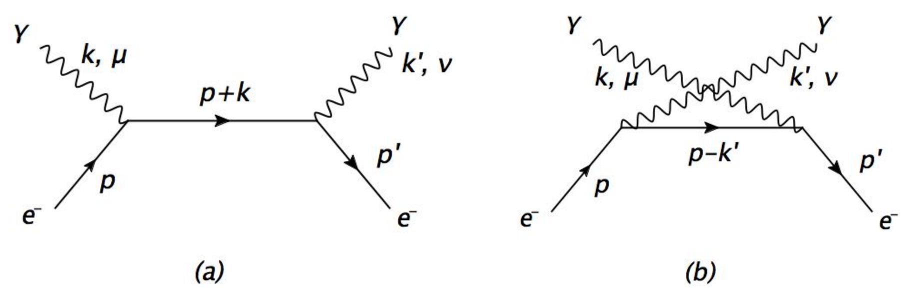

Figure 18.7: Feynman diagrams contributing to Compton scattering at leading order in perturbation theory.

The amplitudes are given by

$$i\mathcal{A}_{(a)} = (-ie)^2\,\epsilon'_\nu\,\bar{u}^{s'}(p')\,\gamma^\nu\,\frac{i}{\not{p}+\not{k}-m}\,\gamma^\mu u^s(p)\,\epsilon^*_\mu\ , \tag{18.58}$$

$$i\mathcal{A}_{(b)} = (-ie)^2\,\epsilon^*_\mu\,\bar{u}^{s'}(p')\,\gamma^\mu\,\frac{i}{\not{p}-\not{k}'-m}\,\gamma^\nu u^s(p)\,\epsilon_\nu\ , \tag{18.59}$$

where the only difference between the two is the contraction of the photon polarizations. Since we need not move fermion fields to obtain one amplitude from the other, this means there is no relative sign between the two contributions. The amplitude square

$$|\mathcal{A}|^2 = |\mathcal{A}_{(a)} + \mathcal{A}_{(b)}|^2 \ , \tag{18.60}$$

has three terms including one of interference. We will first compute $|\mathcal{A}_{(a)}|^2$. For this, let us write the amplitude and its complex conjugate with explicit photon polarization and spinor indices as

$$\begin{aligned} \mathcal{A}_{(a)} &= (-ie)^2\, \epsilon_\nu^{\lambda'}\, \epsilon_\mu^{\lambda *}\, \bar{u}_a^{s'}(p')\, \gamma^\nu_{ab}\, \frac{(\not p + \not k + m)_{bc}}{(p+k)^2 - m^2}\, \gamma^\mu_{cd} u^s_d(p) \\ \mathcal{A}^\dagger_{(a)} &= (-ie)^2\, \epsilon_\beta^{\lambda' *}\, \epsilon_\alpha^{\lambda}\, \bar{u}_e^{s}(p)\, \gamma^\alpha_{ef}\, \frac{(\not p + \not k + m)_{fg}}{(p+k)^2 - m^2}\, \gamma^\beta_{gh} u^{s'}_h(p') \ . \end{aligned} \tag{18.61}$$

To compute the contribution from diagram (a) to the amplitude squared we use the expressions for the sum over the photon polarizations

$$\sum_\lambda \epsilon_\mu^{\lambda *}\, \epsilon_\alpha^{\lambda} = -g_{\mu\alpha} \ , \qquad \sum_\lambda \epsilon_\nu^{\lambda'}\, \epsilon_\beta^{\lambda' *} = -g_{\nu\beta} \ , \tag{18.62}$$

to obtain

$$|\mathcal{A}_{(a)}|^2 = \frac{e^4}{s^2}\, (-g_{\mu\alpha})\, (-g_{\nu\beta})\, u^s_d(p) \bar{u}^s_e(p)\, \gamma^\alpha_{ef}\, (\not p + \not k)_{fg}\, \gamma^\beta_{gh}\, u^{s'}_h(p') \bar{u}^{s'}_a(p')\, \gamma^\nu_{ab}\, (\not p + \not k)_{bc}\, \gamma^\mu_{cd} \ , \tag{18.63}$$

where we took the $m_e \to 0$ limit to further simplify the calculation, and use $s = (p+k)^2$. Finally, summing also over initial and final spinor indices we have

$$\begin{aligned} |\mathcal{A}_{(a)}|^2 &= \frac{e^4}{s^2}\, g_{\mu\alpha}\, g_{\nu\beta}\, \not p_{de}\, \gamma^\alpha_{ef}\, (\not p + \not k)_{fg}\, \gamma^\beta_{gh}\, \not p'_{ha}\, \gamma^\nu_{ab}\, (\not p + \not k)_{bc}\, \gamma^\nu_{cd} \\ &= \frac{e^4}{s^2}\, \mathrm{Tr}\left[\not p \gamma_\mu (\not p + \not k) \gamma_\nu \not p' \gamma^\nu (\not p + \not k) \gamma^\mu\right] \ , \end{aligned} \tag{18.64}$$

The expression above contains the trace over eight gamma matrices. In order to compute it we will use the equality

$$\gamma_\mu \not p \gamma^\mu = -2 \not p \tag{18.65}$$

and similarly for $\not{p}'$. This reduces it to the trace over four gamma matrices. In the $m_e = 0$ approximation the results reads

$$|\mathcal{A}_{(a)}|^2 = 32\frac{e^4}{s^2}\,(p\cdot k)(p'\cdot k)\;, \tag{18.66}$$

Noting that

$$p\cdot k = \frac{s}{2}\;, \qquad p'\cdot k = -\frac{u}{2}\;, \tag{18.67}$$

and dividing by the number of equally probable initial states (2 for the photon and 2 for the electron = 4), we obtain

$$\boxed{\overline{|\mathcal{A}_{(a)}|^2} = 2e^4\,\frac{(-u)}{s}}\;, \tag{18.68}$$

where is worth noting that $u < 0$.

Analogously, we can obtain the diagram (b) contribution, noticing that the denominator from the fermion propagator now goes like u. We have

$$\overline{|\mathcal{A}_{(b)}|^2} = 8\frac{e^4}{u^2}\,(p\cdot k)(p'\cdot k)\;, \tag{18.69}$$

or, in terms of Mandelstam variables,

$$\boxed{\overline{|\mathcal{A}_{(b)}|^2} = 2e^4\,\frac{s}{(-u)}}\;. \tag{18.70}$$

Finally, the interference term is given by

$$\begin{aligned}
\mathcal{A}_{(a)}\mathcal{A}_{(b)}^{\dagger} &= \frac{e^4\,(-g_{\mu\alpha})\,(-g_{\nu\beta})}{(p+k)^2\,(p-k')^2}\,u_h^{s'}(p')\bar{u}_a^{s'}(p')\,\gamma^{\nu}_{ab}\,(\not{p}+\not{k})_{bc}\,\gamma^{\mu}_{cd}\,u_d^{s}(p)\bar{u}_e^{s}(p)\,\gamma^{\beta}_{ef}\,(\not{p}-\not{k}')_{fg}\,\gamma^{\alpha}_{gh} \\
&= \frac{e^4}{s\,u}\,\mathrm{Tr}\left[\not{p}\gamma_{\nu}(\not{p}-\not{k}')\gamma_{\mu}\not{p}'\gamma^{\nu}(\not{p}+\not{k})\gamma^{\mu}\right]\;.
\end{aligned} \tag{18.71}$$

Using that

$$\gamma_{\nu}(\not{p}-\not{k}')\gamma_{\mu}\not{p}'\gamma^{\nu} = -2\,\not{p}'\gamma_{\mu}\,(\not{p}-\not{k}')\;, \tag{18.72}$$

we obtain

$$\mathcal{A}_{(a)}\mathcal{A}^{\dagger}_{(b)} = -2\,\frac{e^4}{s\,u}\,\mathrm{Tr}\left[\not{p}\gamma_\nu(\not{p}-\not{k}')(\not{p}+\not{k})\gamma^\nu\,\not{p}'\right]\;. \tag{18.73}$$

Next, we use the Dirac matrix identity

$$\gamma_\nu\,\not{A}\not{B}\gamma^\nu = 4\,A\cdot B\;, \tag{18.74}$$

so we can rewrite (18.73) as

$$\begin{aligned}\mathcal{A}_{(a)}\mathcal{A}^{\dagger}_{(b)} &= -8\,\frac{e^4}{s\,u}\,(p-k')\cdot(p+k)\,\mathrm{Tr}\left[\not{p}\not{p}'\right]\\ &= -32\,\frac{e^4}{s\,u}\,(p\cdot p')\left(p^2-p\cdot k+k'\cdot p-k'\cdot k\right)\\ &= -16\,\frac{e^4}{s\,u}\,(p\cdot p')\,(s+t+u)\;,\end{aligned} \tag{18.75}$$

where we used that $p^2 = m_e^2 = 0$. But using momentum conservation, i.e. $k' = p+k-p'$, we see that

$$\begin{aligned}s+t+u &= 2p\cdot k - 2p'\cdot p - 2k'\cdot p\\ &= 2(p\cdot k - p'\cdot p - p\cdot k - k^2 + p'\cdot p) = 0\;.\end{aligned} \tag{18.76}$$

Thus, we conclude that

$$\boxed{\mathcal{A}_{(a)}\mathcal{A}^{\dagger}_{(b)} = 0}\;. \tag{18.77}$$

The intereference term in general will not vanish for finite m_e, but it is greatly suppressed at very high energies. Then, in this limit the amplitude squared for Compton scattering is the sum of (18.68) and (18.70)

$$\boxed{\overline{|\mathcal{A}|^2} = 2e^4\left[\frac{(-u)}{s}+\frac{s}{(-u)}\right]}\;. \tag{18.78}$$

18.6 $e^-e^+ \to \gamma\gamma$

Electron–positron annihilation into photons is a process related to Compton scattering by crossing symmetry. The corresponding diagrams are depicted in Figure 18.8.

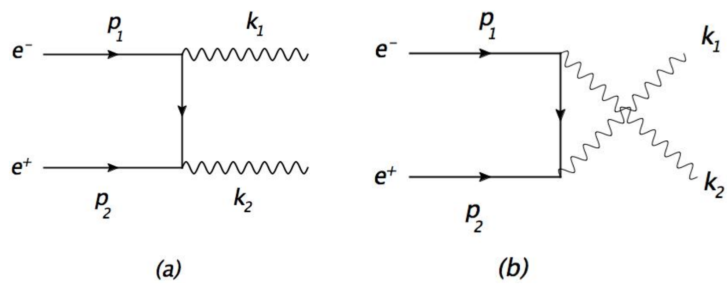

Figure 18.8: Feynman diagrams contributing to $e^-e^+ \to \gamma\gamma$ at leading order in perturbation theory.

Comparing the Feynman diagrams from Figures 18.7 and 18.8, we can obtain the amplitude squared for this process from the one for Compton scattering by making the following replacements:

$$\begin{array}{ccr} p & \longrightarrow & p_1 \\ p' & \longrightarrow & -p_2 \\ k' & \longrightarrow & k_2 \\ k & \longrightarrow & -k_1 \end{array} \tag{18.79}$$

This results in the following change in Mandelstam variables

$$\begin{array}{rcr} s = (p+k)^2 & \longrightarrow & (p_1 - k_1)^2 = t \\ t = (k'-k)^2 & \longrightarrow & (k_2 + k_1)^2 = s \\ u = (k'-p)^2 & \longrightarrow & (k_2 - p_1)^2 = u\ . \end{array} \tag{18.80}$$

Then the amplitude squared for $e^-e^+ \to \gamma\gamma$ in the $m_e = 0$ approximation is vs

$$\overline{|\mathcal{A}|^2} = 2e^4 \left[\frac{(-u)}{t} + \frac{t}{(-u)} \right]\ . \tag{18.81}$$

It is straightforward to see that the amplitude for $\gamma\gamma \to e^-\,e^+$ can be obtained directly from this one.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 5.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 5.