Lecture 17

Gauge Theories in the Functional Integral

As we saw in the previous lecture, the quantization of gauge fields is more subtle than what we anticipated. The main reason is gauge invariance, or the redundancy that it brings. The problem is more clearly identified in the functional integral formulation. In this part of the course we will restrict ourselves to abelian gauge theories, or $U(1)$ gauge theories. But the functional integral approach will come in handy when quantizing non-abelian gauge theories.

17.1 Gauge Redundancy in the Functional Integral

The action for the gauge fields in a $U(1)$ gauge theory is

$$S[A_\mu] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\ , \tag{17.1}$$

where as usual we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ , \tag{17.2}$$

is the gauge invariant gauge tensor. Writing the action out in terms of A_μ and its derivatives we have

$$\begin{aligned} S[A_\mu] &= -\frac{1}{2}\int d^4x\, \{\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\nu\} \\ &= \frac{1}{2}\int d^4x\, A_\nu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\mu\ , \end{aligned} \tag{17.3}$$

where to obtain the last equality we integrated by parts both terms in the first equality. So we can write the action as a quadratic form

$$S[A_\mu] = \frac{1}{2}\int d^4x\, A_\nu\, \mathcal{O}^{\mu\nu}\, A_\mu \ , \tag{17.4}$$

with

$$\mathcal{O}^{\mu\nu} \equiv g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \ . \tag{17.5}$$

This means that we can use our usual trick to obtain the generating functional of the theory. The generating functional in the presence of a source linearly coupled to A_μ is

$$Z[J_\mu] = N\int \mathcal{D}A_\mu e^{iS[A_\mu]+i\int d^4x\, J_\mu(x)A^\mu(x)} \ . \tag{17.6}$$

Then we know that the fact that the action is quadratic in the fields allows us to make a change of variables in $A_\mu(x)$ in order to decouple the source $J_\mu(x)$. The answer for the generating functional should be

$$Z[J_\mu] = Z[0]\, e^{-1/2\int d^4x\, d^4y\, J^\mu(x) D_{F\mu\nu}(x-y)\, J^\nu(y)} \ , \tag{17.7}$$

where the two-point function $D_{F\mu\nu}(x-y)$ must be a Green function of the operator $\mathcal{O}^{\mu\nu}$, i.e it must satisfy

$$\mathcal{O}^{\mu\nu}_x\, D_{F\nu\rho}(x-y) = i\delta^{(4)}(x-y)\delta^\mu_\rho \ . \tag{17.8}$$

This appears to solve the problem. Just deriving twice the expression above we can obtain the gauge boson propagator, i.e. $D_{F\mu\nu}(x-y)$. However, it turns out that this does not work since $\mathcal{O}^{\mu\nu}$ does not have an inverse. To see this we observe that taking a derivative on $\mathcal{O}^{\mu\nu}$ results in

$$\begin{aligned}\partial_{\mu x}\mathcal{O}^{\mu\nu}_x D_{F\nu\rho}(x-y) &= \partial_{\mu x}\left(g^{\mu\nu}\partial^2_x - \partial^\mu_x\partial^\nu_x\right) D_{F\nu\rho}(x-y) \\ &= 0\times D_{F\nu\rho}(x-y) = \partial_{\mu x} i\delta^{(4)}(x-y)\delta^\mu_\rho \ ,\end{aligned} \tag{17.9}$$

which can only be satisfied if $D_{F\nu\rho}(x-y)$ is infinite. Another way to see what is happening is to consider an arbitrary function $\alpha(x)$. Then we can see that

$$\mathcal{O}^{\mu\nu}\,\partial_\nu\alpha(x) = 0\ , \tag{17.10}$$

which means that $\partial_\nu\alpha(x)$ is an eigenfunction of $\mathcal{O}^{\mu\nu}$ with zero eigenvalue. Once again, this means that $\mathcal{O}^{\mu\nu}$ cannot be inverted.

The origin of the problem is gauge invariance. Under a gauge transformation, the gauge field changes as

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\,\partial_\mu\alpha(x)\ , \tag{17.11}$$

for an arbitrary function $\alpha(x)$, leaving the action invariant. That is

$$S[A'_\mu(x)] = S[A_\mu(x)]\ . \tag{17.12}$$

As we will see below, this poses a problem since the functional integration treats $A_\mu(x)$ and $A'_\mu(x)$ as two distinct field configurations that must be independently considered. However this is not correct since the action is the same for both configurations. That is field configurations that are related by gauge transformations are not physically distinct and should not be counted separately in the functional integration.

To proceed we will divide the field configurations into two: those not connected by gauge transformations, call them $\bar{A}_\mu(x)$, and those that can be reach from these by a gauge transformation characterized by a function $\alpha(x)$. Thus, the function integral over all the gauge field configurations can be written as

$$\int \mathcal{D}A_\mu e^{iS[A_\mu]} = \int \mathcal{D}\bar{A}_\mu\, e^{iS[\bar{A}_\mu]} \int \mathcal{D}\alpha\ , \tag{17.13}$$

where we have split the gauge field configurations by integrating over the inequivalent gauge fields $\bar{A}_\mu$ and over all the possible functions $\alpha(x)$ that can be used to generate other configurations from them. We can see in (17.13) that since the action is invariant under gauge transformations, these will not generate additional terms in the exponential. This means that the integration over the $\alpha(x)$ is not exponentially regulated. In other words, the gauge field configurations that can be accessed via gauge transformations are not only physically equivalent to the original ones, but they result in unweighted contributions to the functional integral. The integral over $\alpha(x)$ is infinite, or not well defined. In order to properly quantize the gauge theory we will need to only count the inequivalent gauge field configurations $\bar{A}_\mu(x)$.

17.2 The Fadeev-Popov Method

To separate the unphysical degrees of freedom accessible through gauge transformations we will use a trick due to Fadeev and Popov. First, we will consider an example with a discrete set of integration variables in order to introduce the method. Let us consider an $N \times N$ matrix $\mathbf{A}$, and define the functional

$$Z[\mathbf{A}] = \left[\prod_{i=1}^{N} \int_{-\infty}^{+\infty} dx_i\right] e^{-\sum_{j,k} x_j A_{jk} x_k} \;. \tag{17.14}$$

The matrix $\mathbf{A}$ can be diagonalized by a linear transformation $\mathbf{R}$ as in

$$\mathbf{A}^{\text{diag.}} = \mathbf{R}\,\mathbf{A}\mathbf{R}^{-1} \;, \tag{17.15}$$

where $\mathbf{R}$ defines new coordinates by

$$\begin{aligned} \mathbf{y} &= \mathbf{R}\,\mathbf{x} \\ &\text{or} \\ y_i &= R_{ij}\,x_j \end{aligned} \tag{17.16}$$

where in the last equality we assumed that repeated indexes as summed over. Then, the functional in (17.14) can be written as

$$\begin{aligned} Z[\mathbf{A}] &= \left[\prod_{i=1}^{N} \int_{-\infty}^{+\infty} dy_i\right] e^{-\sum_{k,\ell} y_k A^{\text{diag.}}_{k\ell} y_\ell} \\ &= \left[\prod_{i=1}^{N} \int_{-\infty}^{+\infty} dy_i\right] e^{-\sum_{k} y_k^2 A^{\text{diag.}}_{kk}} \\ &= \prod_{i=1}^{N} \left(\frac{\pi}{A^{\text{diag.}}_{ii}}\right)^{1/2} \\ &= \pi^{N/2}\left[\det\mathbf{A}\right]^{-1/2} \;, \end{aligned} \tag{17.17}$$

where we have used the fact that $\det[\mathbf{R}] = 1$.

Now let us assume that some of the eigenvalues of $\mathbf{A}$ are zero. Clearly this is analogous the situation in the gauge field functional integral where the operator $\mathcal{O}^{\mu\nu}$ had eigenfunctions of zero eigenvalues as seen in (17.10). For convenience, let us assume that the last n eigenvalues of $\mathbf{A}$ vanish. That is

$$A^{\text{diag.}}_{kk} = 0, \quad \forall \quad N-n+1 \leq k \leq N \;, \tag{17.18}$$

which means that we can rewrite (17.17) as

$$Z[\mathbf{A}] = \left[\prod_{i=1}^{N-n} \int_{-\infty}^{+\infty} dy_i e^{-y_i^2 A^{\text{diag.}}_{ii}}\right] \times \left[\prod_{j=N-n+1}^{N} \int_{-\infty}^{+\infty} dy_i\right] \;, \tag{17.19}$$

where to get the second factor we used (17.18). The expression (17.19) is analogous to (17.13) in the function integral for gauge fields. The integrals in the second factor of this expression are all divergent as they are not suppressed by exponentials. This divergences are also reflected in the fact that

$$\det\mathbf{A} \longrightarrow 0 \;, \tag{17.20}$$

in (17.17) due to the presence of zero eigenvalues. In this example is trivial to separate the finite part of the functional integral by restricting ourselves to the non-vanishing eigenvalues. That is the finite part of the functional integral in (17.19) is simply

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N-n} \int_{-\infty}^{+\infty} dy_i e^{-y_i^2 A^{\text{diag.}}_{ii}}\right] \;, \tag{17.21}$$

Since this is analogous to the first factor in the functional integral (17.13), the integral over the inequivalent and physically relevant field configurations, it would be interesting to try to write down (17.21) in a way that did not imply an apriori separation of $\bar{A}_\mu$ from the gauge redundant field configurations. To implement this in our matrix example would mean to integrate over all the indices from $i = 1$ to N, but somehow avoiding the infinities. For this purpose we define new variables z_i by

$$z_i = \begin{cases} y_i, & \forall \quad 1 \leq i \leq N-n \\ \text{anything} & \forall \quad N-n+1 \leq i \leq N \end{cases} \tag{17.22}$$

such that we can write the finite part of the functional integral as an integral over all the indices, even those corresponding to zero $\mathbf{A}$ eigenvalues. To do this we write

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N} \int dz_i\right] \delta(z_{N-n+1})\ldots\delta(z_{N-1})\delta(z_N)e^{-\sum_{k,\ell} z_k A_{k\ell} z_\ell} , \tag{17.23}$$

where the δ functions in (17.23) turn the infinite integrals of the second factor in (17.19) into unity. In this way we selected the finite part of the integral only, but we are still, at least formally, integrating over all the variables. As a last step we can go back to our original variables, x_i, in which $\mathbf{A}$ was generically non-diagonal. Then the finite functional integral can be finally written as

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N} \int dx_i\right] \det\left|\frac{\partial z_i}{\partial x_j}\right| \prod_{j=N-n+1} \delta(z_j(\mathbf{x}))\, e^{-\sum_{k,\ell} x_k A_{k\ell} x_\ell} , \tag{17.24}$$

where the factor of

$$\det\left|\frac{\partial z_i}{\partial x_j}\right| , \tag{17.25}$$

corresponds to the Jacobian of the transformation from the $\mathbf{z}$ to the $\mathbf{x}$ variables.

We can apply the result of (17.24) to our functional integral over the gauge fields (17.13). Since we only want to compute the integral over the gauge inequivalent fields, i.e. the first factor in (17.13), we write

$$\boxed{Z^f[A_\mu] = \int \mathcal{D}A_\mu \, \det\left[\frac{\delta G}{\delta\alpha}\right] \, \delta\left(G[A_\mu]\right) \, e^{iS[A_\mu]}} . \tag{17.26}$$

Here the $\alpha(x)$ are the parameters of the gauge transformations, and $G[A_\mu(x)]$ are functionals that vanish for a certain type of configuration $A_\mu(x)$. These are called gauge fixing functionals and they have the role of removing the gauge redundancy. The functional derivative

$$\frac{\delta G[A_\mu(x)]}{\delta\alpha(x)} , \tag{17.27}$$

gives the response of $G[A_\mu(x)]$ to changes in the gauge parameter function $\alpha(x)$. Its determinant is analogous to the Jacobian of the $\mathbf{z} \to \mathbf{x}$ transformation in our matrix example.

To see how this works let us start with a familiar example of gauge fixing. We choose

$$G[A_\mu] = \partial_\mu A^\mu - \kappa \ , \tag{17.28}$$

with κ an arbitrary constant. Under the gauge transformation

$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha \ , \tag{17.29}$$

we have

$$G[A_\mu] \longrightarrow \partial_\mu A^\mu + \frac{1}{e}\partial^2 \alpha - \kappa \ , \tag{17.30}$$

which results in

$$\frac{\delta G[A_\mu]}{\delta \alpha} = \frac{1}{e}\,\partial^2 \ . \tag{17.31}$$

We see that the functional derivative giving the response of the gauge fixing functional is an operator that does not depend on the gauge field $A_\mu(x)$. Independently of how to evaluate its determinant, the important point is that it can be taken outside the functional integral (17.26) so as to become part of the normalization. Since the normalization of the generating functional is not important, then we see that the determinant in this case is innocuos. This is always the case in abelian gauge theories. As we will see later in the course, non-abelian gauge theories will have additional dependence on the gauge fields appearing in the gauge transformation. This will result in a non-trivial dependence of the determinant in (17.26) on the gauge fields being integrated, and in the need to introduce new (albeit non-physical) degrees of freedom in the quantization of non-abelian gauge theories. But for now, we do not have to worry about the determinant.

Then, we can write down the generating functional over the gauge fields in the presence of a linearly coupled vector source as

$$Z[J_\mu] = N \int \mathcal{D}A_\mu \ \delta\left(\partial_\mu A^\mu - \kappa\right) \ e^{i\int d^4x \left\{1/2 A_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu\right) A_\nu + J_\mu A^\mu\right\}} \ , \tag{17.32}$$

where we have written out the operator $\mathcal{O}^{\mu\nu}$ explicitly in order to apply the condition in the delta function. Once this is done we obtain

$$Z[J_\mu] = N \int \mathcal{D}A_\mu \ e^{i\int d^4x \left\{1/2 A_\mu g^{\mu\nu}\partial^2 A_\nu + J_\mu A^\mu\right\}} \ , \tag{17.33}$$

where using the condition inside the delta function eliminates the second term of $\mathcal{O}^{\mu\nu}$. Now we can write our usual solution,

$$Z[J_\mu] = Z[0]\, e^{-\frac{1}{2}\int d^4x d^4y J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y)} \;, \tag{17.34}$$

where now $D_F^{\mu\nu}(x-y)$ must be a Green function of the Dalembertian operator $g^{\mu\nu}\partial^2$. That is

$$g^{\mu\nu}\partial_x^2 D_{F\nu\rho}(x-y) = i\delta^{(4)}(x-y)\,\delta^\mu_\rho \;,. \tag{17.35}$$

In momentum space, the solution is clearly given by

$$D_{F\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-ig_{\mu\nu}}{k^2+i\epsilon} \;. \tag{17.36}$$

This is then the gauge boson propagator with this particular choice of gauge fixing function. It is called the Feynman gauge.

More generally, we can assume that the gauge fixing condition is

$$G[A_\mu(x)] = \partial_\mu A^\mu - \kappa(x) \;, \tag{17.37}$$

that is elevating κ to a function. We can think of $\kappa(x)$ as if it was a field to be integrated over in the functional integral. If we were to introduce $\kappa(x)$, we would add a quadratic term on it so as to give just an innocuos shift in the normalization. Something like

$$\int \mathcal{D}\kappa\; e^{-i\int d^4x \frac{\kappa^2(x)}{2\xi}} \;, \tag{17.38}$$

where ξ is an arbitrary constant. But in our case, since there is a delta function involving $\kappa(x)$, we will have

$$\begin{aligned} Z[J_\mu] &= N\int \mathcal{D}A_\mu \mathcal{D}\kappa\, \delta(\partial_\mu A^\mu - \kappa(x))\, e^{i\int d^4x \left\{1/2 A_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial\nu\right)A_\nu - (1/2\xi)\kappa^2(x) + J_\mu(x)A^\mu(x)\right\}} \\ &= N\int \mathcal{D}A_\mu e^{i\int d^4x\left\{1/2A_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial\nu\right)A_\nu - (1/2\xi)(\partial_\mu A^\mu)^2 + J_\mu(x)A^\mu(x)\right\}} \;. \end{aligned} \tag{17.39}$$

Again we have a quadratic form that we can integrate making the usual change of variables in $A_\mu(x)$. The operator in the quadratic form in the gauge fields is now

$$\tilde{\mathcal{O}}^{\mu\nu} = g^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \;, \tag{17.40}$$

where we integrated by parts the term depending on ξ in order to have the desired quadratic form. In this way, if we now write (17.34) for the solution for the generating functional, the two-point function in it must be a Green function of $\tilde{\mathcal{O}}^{\mu\nu}$, i.e.

$$\tilde{\mathcal{O}}^{\mu\nu}_x D_{F\nu\rho}(x-y) = i\delta^{(4)}(x-y)\,\delta^{\mu}_{\rho} \ . \tag{17.41}$$

If we express the two-point function in terms of its Fourier transform $\hat{D}_{F\nu\rho}(k)$ as

$$D_{F\nu\rho}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\hat{D}_{F\nu\rho}(k)\,e^{-ik\cdot(x-y)} \ , \tag{17.42}$$

Applying the operator in the integrand in (17.42) we obtain

$$\left[-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^{\mu}k^{\nu}\right]\hat{D}_{F\nu\rho}(k) = i\delta^{\mu}_{\rho} \ . \tag{17.43}$$

In order to invert the above expression and obtain the propagator in momentum space we write the most general form it can have as a function of momentum. This is

$$\hat{D}_{F\nu\rho}(k) = A(k^2)g_{\nu\rho} + B(k^2)k_{\nu}k_{\rho} \ , \tag{17.44}$$

where the coefficients A and B are functions of the only Lorentz invariant, k^2. Inserting this in (17.43) and solving for the coefficients we arrive at

$$\boxed{\hat{D}_{F\nu\rho}(k) = -\frac{i}{k^2}\left[g_{\nu\rho} - (1-\xi)\,\frac{k_{\nu}k_{\rho}}{k^2}\right]} \ . \tag{17.45}$$

This is the gauge boson propagator in the so-called R_ξ gauge, for arbitrary values of ξ. Choosing ξ we fix the gauge. For instance, with the $\xi = 1$ we obtain the propagator of (17.36). But in many cases other choices may be more convenient. The choice $\xi = 0$ is called the Landau gauge. In general we will be using $\xi = 1$.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 9.4.

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 3.2

- *Quantum Field Theory* , by M. Srednicki, Chapter 58.