Lecture 16

Gauge Fields

In this section we will introduce vector fields. Although it is generically possible to write the action for a theory with such fields, it turns out that these generic theories are not well defined unless the vector fields are gauge fields, i.e. vector fields associated with a local symmetry. We will eventually show this relation further along our course. For now, let us introduce gauge fields as a consequence of gauge invariance. We will start with a fermion theory so as to derive quantum electrodynamics.

16.1 Gauge Invariance

Let us consider the lagrangian for a free fermion of mass m

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \ . \tag{16.1}$$

This is invariant under the *global* $U(1)$ transformation[1] defined by

$$\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha}\,\psi(x) \ , \\ \bar{\psi}(x) &\longrightarrow e^{-i\alpha}\,\bar{\psi}(x) \ , \end{aligned} \tag{16.2}$$

where α is a real constant. The conserved charge associated with these symmetry transformations is fermion number: $+1$ for fermions, -1 for antifermions.

But what if we want *local* $U(1)$ invariance, i.e. what if $\alpha = \alpha(x)$ is a function of the spacetime position ? The local transformation now reads

[1] A unitary transformation determined by one parameter.

$$\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha(x)}\,\psi(x)\ , \\ \bar{\psi}(x) &\longrightarrow e^{-i\alpha(x)}\,\bar{\psi}(x)\ , \end{aligned} \tag{16.3}$$

which leads to a transformation of the lagrangian as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \partial_\mu\alpha(x)\,\bar{\psi}\gamma^\mu\psi\ \neq \mathcal{L}\ . \tag{16.4}$$

From (16.4) we see that the local or *gauge* transformation (16.3) does not leave the lagrangian (16.1) invariant. In order to obtain a theory invariant under these local transformations we will need to add a new field that also transforms in some way that depends on $\alpha(x)$ and whose transformation cancels the extra term that appears in (16.4). One way to do this is to define a covariant derivative on $\psi(x)$, a generalization of the normal derivative. We write

$$\mathcal{L} = \bar{\psi}\,(iD\!\!\!\!/\, - m)\,\psi\ , \tag{16.5}$$

where we defined the covariant derivative $D_\mu\psi(x)$ so that it must transform as the field $\psi(x)$ in order for (16.5) to be invariant, i.e. under the transformations (16.3) it must transform as

$$D_\mu\psi(x) \longrightarrow e^{i\alpha(x)}\,D_\mu\psi(x)\ . \tag{16.6}$$

Clearly, we can see that if (16.6) is satisfied at the same time as (16.3) then (16.5) is invariant. Next, we write the covariant derivative $D_\mu\psi(x)$ by introducing a vector field as

$$D_\mu\psi(x) \equiv \left(\partial_\mu + ieA_\mu(x)\right)\,\psi(x)\ , \tag{16.7}$$

where e is a constant. Then, it can be verified that in order for the covariant derivative defined in (16.7) to satisfy (16.6) the vector field $A_\mu(x)$ must transform as

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu\alpha(x)\ . \tag{16.8}$$

We notice in passing that the vector field $A^\mu(x)$ must be real. This is a consequence of the fact that the gauge parameter $\alpha(x)$ is real. Thus, to summarize, the theory in (16.5) is invariant under the gauge or local $U(1)$ transformations

$$\begin{aligned}
\psi(x) &\longrightarrow e^{i\alpha(x)}\,\psi(x)\ , \\
\bar{\psi}(x) &\longrightarrow e^{-i\alpha(x)}\,\bar{\psi}(x)\ , \\
A_\mu(x) &\longrightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu\alpha(x)\ ,
\end{aligned} \tag{16.9}$$

with the covariant derivative defined by (16.7). Finally, if the gauge field $A_\mu(x)$ is to be a dynamical degree of freedom, we need appropriate quadratic terms in it, i.e. a kinetic term and a mass term. A kinetic term that is trivially invariant under the transformations (16.8) is built from the contraction of

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\ , \tag{16.10}$$

with itself, since the tensor $F_{\mu\nu}$ is invariant. Furthermore, a mass term for $A_\mu(x)$ must be something like

$$m_A^2\,A_\mu A^\mu\ . \tag{16.11}$$

But since this is clearly not gauge invariant, we must assume that $m_A = 0$. Thus a gauge field must have zero mass in order to respect gauge invariance. Although there are exceptions to this statement, they all correspond to the case when the mass is generated dynamically via a scalar field coupled to $A_\mu(x)$ obtaining a non-zero vacuum expectation value. We will study this case in the second part of this course. For now, gauge invariance means zero mass for the gauge fields. Then, the complete theory that is $U(1)$ gauge invariant is

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \bar{\psi}\,(i\not{\partial} - m)\,\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ ,
\end{aligned} \tag{16.12}$$

where in the last equality we can see that the gauge field $A_\mu(x)$ interacts with the fermion current with a coupling e. The factor of $-1/4$ in front of the gauge field kinetic term is a convenient choice of normalization which results in $F_{\mu\nu}$ being the electromagnetic stress tensor in the case of quantum electrodynamics (QED). In fact, this lagrangian is the basis for QED, where $\psi(x)$ is the charged electron field and $A_\mu(x)$ is identified with the photon. The next step in order to obtain QED as a quantum field theory would be to quantize the gauge field $A_\mu(x)$.

16.2 Gauge Fields and Quantization

From the lagrangian for the gauge fields

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \tag{16.13}$$

we can derive the equations of motion (Euler-Lagrange)

$$\partial^2 A^\mu - \partial^\mu \left(\partial_\nu A^\nu \right) = 0 \ , \tag{16.14}$$

As usual in the classical case, if we choose the Lorentz condition

$$\partial_\mu A^\mu = 0 \ , \tag{16.15}$$

we obtain

$$\partial^2 A^\mu = 0 \ . \tag{16.16}$$

Thus, imposing the Lorentz condition (16.15) gives us a simple equation with plane wave solutions, a massless Klein-Gordon equation for each component of the four-vector $A^\mu(x)$. Naively, we would then expand $A^\mu(x)$ in these solutions and quantize by imposing commutation relations between $A^\mu(x)$ and its conjugate momentum $\pi^\mu(x)$. From (16.13) we obtain the form of the conjugate momentum as

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} \ . \tag{16.17}$$

However, from (16.17) it is clear that there is a problem with the time component of $\pi^\mu(x)$ coming from the fact that $F^{\mu\nu}$ is antisymmetric. We have that

$$\pi^0(x) = 0 \ , \tag{16.18}$$

meaning that it will not be possible to impose a quantization condition on $A^0(x)$. We can get around this by adding a term to the lagrangian as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - c(\partial_\mu A^\mu)^2 \ , \tag{16.19}$$

where c is a arbitrary real constant. Now the equations of motion are

$$\partial^2 A^\mu + (c-1)\partial^\mu(\partial_\nu A^\nu) = 0 \ . \tag{16.20}$$

We can see that there are two ways of obtaining (16.16): either by using the Lorentz condition or by choosing $c = 1$. But now the second choice also allows us to define a non-zero conjugate momentum of $A^0(x)$ since now

$$\pi^\mu(x) = F^{\mu 0} - c g^{\mu 0}(\partial_\nu A^\nu) \ , \tag{16.21}$$

which results in

$$\pi^0(x) = -c(\partial_\nu A^\nu) \ . \tag{16.22}$$

So choosing $c = 1$ allows us to carry out the canonical quantization procedure. This is called the Feynman gauge. But, as we will see below, the upshot is that now we will have non-physical degrees of freedom.

To proceed with the quantization, we start by expanding the field $A^\mu(x)$ in momentum space. The most general solution to (16.16) can be written as

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \sum_{\lambda=0}^{3} \left\{ a_k^{(\lambda)} \epsilon_\mu^{(\lambda)} e^{-ik\cdot x} + a_k^{\dagger(\lambda)} \epsilon_\mu^{*(\lambda)} e^{+ik\cdot x} \right\} \ , \tag{16.23}$$

where we have used the fact that $A^\mu(x)$ must be a real field, and the $\epsilon_\mu^{(\lambda)}$ for $\lambda = 0, 1, 2, 3$ form a basis for a general expansion of any four-vector, the so-called polarization vectors. If we could use the Lorentz condition (16.15) we could eliminate one of the polarizations through

$$k^\mu \epsilon_\mu^{(\lambda)} = 0 \ , \tag{16.24}$$

In particular, using gauge invariance we can always eliminate the polarization with time components. This is desirable for the quantization procedure given that in its presence there appear negative norm states. To see this let us guess the form of $\langle 0|T A_\mu(x) A_\nu(y)|0\rangle$, which should be the gauge boson propagator. Since each component of $A_\mu(x)$ obeys the massless Klein-Gordon equation all we lack to write it is to guess its tensor form: it should be an isotropic second rank tensor. Let us try $g_{\mu\nu}$. We then write

$$\langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \int \frac{d^4q}{(2\pi)^4} \frac{-i g_{\mu\nu}}{q^2 + i\epsilon} e^{-iq\cdot(x-y)} \ . \tag{16.25}$$

To understand the sign choice we notice that doing the contour integral in q_0 we obtain

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^3q}{(2\pi)^3}\,\frac{-g_{\mu\nu}}{2E_q}\,e^{-iq\cdot(x-y)} \ . \tag{16.26}$$

If we now take $x \to y$ (but with the limit $x_0 \to y_0$ from the positive side) and take $\mu = \nu$, then the quantity in (16.26) becomes the norm of the state

$$A_\mu(x)|0\rangle \ . \tag{16.27}$$

We want states associated with the physical polarizations of real photons, which must be spatial in nature, e.g. $A_i(x)$ for $i = 1, 2$, to have positive norm. This forces us to choose the minus sign in front of $g_{\mu\nu}$ in (16.25). But at the same time this means that the sate

$$A_0(x)|0\rangle \ , \tag{16.28}$$

must have negative norm. This sounds troublesome. Although we will see later that these states are not really produced in physical processes, they do complicate the quantization procedure. For instance, it is possible to work around this by making specific gauge choices, but they all break Lorentz invariance. We will take a different approach and derive the photon propagator using functional methods.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 4.8.
- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 3.2
- *The Quantum Theory of Fields, Vol. I*, by S. Weinberg, Section 8.1 to 8.3.