Lecture 15

Cross Sections: A Simple Example in Yukawa Theory

We will complete the path from a lagrangian to observables by computing the cross section for a spacific process in Yukawa theory. We will choose an example that minimizes the amount of computation in favor of showing the important steps of the calculation. For this purpose we will assume that there are two different fermions, χ and ψ, with $m_\chi \gg m_\psi$. We will the assume that the lagrangian is

$$\begin{aligned} \mathcal{L} &= \bar{\chi}(i\,\partial\!\!\!/ - m_\chi)\chi + \bar{\psi}(i\,\partial\!\!\!/ - m_\psi)\psi - g\bar{\chi}\chi\phi - g\bar{\psi}\psi\phi \\ &\quad + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\mu^2\phi^2 + \dots \end{aligned} \tag{15.1}$$

where the dots denote terms depending on powers of ϕ such as the tadpole, or ϕ^3 or ϕ^4, which are inevitably generated by fermion loops. But we will not make use of any of them in this chapter given that we will work in lowest order in perturbation theory. But before we go on and compute the cross section, we will digress and discuss some general aspects of lagrangians and dimensions.

15.1 Dimensional Analysis (*Digression*)

Since the Yukawa lagrangian in (15.1) is full of elements (fermion and scalar fields, couplings in interaction terms) this is a good time for us to pause and talk about units (yes, units!). It turns out that dimensional analysis is a very helpful tool in quantum field theory. This will be starkly clear when we start studying renormalization. In the meantime, we introduce a few of the concepts here since at the end of this lecture we will already see some of their consequences in action. We start with the simple observation that the action S has no dimensions (remember here we are using the unit system with $\hbar = 1$ and $c = 1$). Thus, since

$$S = \int d^4x \, \mathcal{L} \ , \tag{15.2}$$

we see that *every term* in the lagrangian density $\mathcal{L}$ must have units of length to the minus fourth power or, alternatively, energy to the fourth power:

$$[\mathcal{L}] = L^{-4} = E^4 \ , \tag{15.3}$$

where the squared brakets above return the units of the argument. Let us consider the lagrangian in (15.1). In order to understand the units of every element in it, we start with the one parameter we know: the mass of any given field must have units of energy. So we have that, since $[\mu] = [m_\chi] = [\mu_\psi] = E$, we have that

$$[\phi] = E \ , \qquad [\psi] = [\chi] = E^{3/2} \ . \tag{15.4}$$

As you can see, this is a general result: scalar fields have dimension 1 and fermions dimension 3/2 in units of energy, so that their mass terms have the appropriate dimension four. We see that kinetic terms are also consistent with this assignments since derivatives have also dimension one:

$$[\partial_\mu] = [\frac{\partial}{\partial x^\mu}] = L^{-1} = E \ . \tag{15.5}$$

Armed with this information we can now look at the interaction terms in any lagrangian. For instance, an interaction term such as

$$-\frac{\lambda}{4!}\phi^4 \ , \tag{15.6}$$

that we have been considering, corresponds to the coupling λ being *dimensionless*, i.e.

$$[\lambda] = E^0 \ . \tag{15.7}$$

Similarly, if we consider the fermion-scalar coupling $-g\phi\bar{\psi}\psi$, we conclude that

$$[g] = E^0, \tag{15.8}$$

i.e. g is also dimensionless. We see that the couplings that correspond to coefficients of terms in the lagrangian of dimension four, are always dimensionless, whereas if the terms have dimensions smaller than four, the coefficient will have *positive* energy dimensions

(e.g. the mass term), and finally if the term in question has dimensions greater than four its coefficient will have *negative* energy dimensions. An example of one this would be a four-fermion operator such as

$$\bar{\psi}\psi\,\bar{\psi}\psi\ , \qquad \bar{\chi}\chi\,\bar{\chi}\chi\ , \qquad \bar{\psi}\psi\,\bar{\chi}\chi\ . \tag{15.9}$$

These operators are of dimension six. We could imagine obtaining them from (15.1) if we consider the case with $\mu \gg m_\psi, m_\chi$ and much larger than the energy of the processes being considered. Then we can imagine integrating out the field ϕ which would result in the operators in (15.9) having a coefficient given by

$$-\frac{g^2}{\mu^2}\ , \tag{15.10}$$

which, as advertised, has energy dimensions E^{-2}. In general, the coefficient of a given operator of dimensions d_O in the lagrangian has energy dimensions given by

$$E^{4-d_O}\ . \tag{15.11}$$

As we will see in detail in a few lectures, the dimension of the operator in the lagrangian determines whether or not this is an operator that is *renormalizable* (dimension four or dimensionless coefficient), super-renormalizable (dimension less than four or coefficient with positive energy dimensions) ou *non-renormalizable* (dimension greater than four or coefficient of negative energy dimensions). A term in the lagrangian is deemed renormalizable if, when we consider the effects of loop corrections generated by this term, they generate a *finite* number of distinct divergencies. This will mean that these can be absorbed by the redefinition of a finite number of parameters of the theory.

In contrast, terms with dimensions greater than four will require an *infinite* number of redefinitions, i.e. an infinite number of parameters. This does not mean that theories containing non-renormalizable terms are of no use. But they are of limited use, where the limitation is to low energies. In particular, we can ignore non-renormalizable terms in a lagrangian if we imagine that the energy scales suppressing the coefficients of these higher-dimensional operators are very large compared with the scales of interest. In this way, we can concentrate in a finite number of terms in $\mathcal{L}$. There are infinitely many non-renormalizable terms we can write, but by ignoring them we are assuming that their effects are negligible compared to the ones coming from the renormalizable operators. We will discuss all of these questions in great detail in a few lectures. However, when computing cross sections of some processes using (15.1), we will be able to already see the consequences of non-renormalizability at high energies through the violation of unitarity.

15.2 Computing the Amplitude

Going back to the theory described by (15.1), we would like to compute the cross section for the scattering process $\chi\bar{\chi} \to \psi\bar{\psi}$, to leading order in perturbation theory. This will not require that we consider any of the ϕ-dependent terms denoted in dots in (15.1). In order to obtain the amplitude for this process we will have to first draw all possible Feynman diagrams. It turns put that there is only one and it is shown in Figure 15.1.

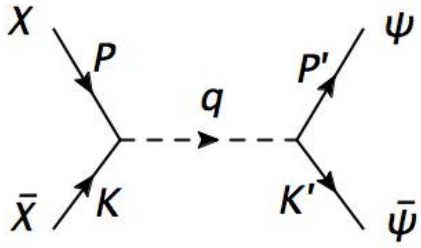

Figure 15.1: Feynman diagram in momentum space contributing to $\chi\bar{\chi} \to \psi\bar{\psi}$ in lowest order in g in the theory described by the lagrangian in (15.1).

The reason why this is the only diagram to this order is that the lagrangian in (15.1) respects two separate global $U(1)$ symmetries: one for χ and another one for ψ. That is

$$\begin{aligned} \chi &\to e^{i\alpha}\chi \,, \\ \psi &\to e^{i\beta}\psi \,, \end{aligned}$$

where α and β are independent constants. This resutls in two separately conserved charges Q_χ and Q_ψ which forbid the vertices of the type $\bar{\chi}\psi\phi$ that violate both charge conservations.

15.3 Squaring the Amplitude

Going back to the diagram of Figure 15.1, we can write the amplitude using the Feynman rules for Yukawa theory derived in the previous chapter. The amplitude is given by

$$i\mathcal{A} = \bar{u}_c^{r'}(p')v_c^{s'}(k')\,(-ig)\,\frac{i}{q^2-\mu^2}\,(-ig)\,\bar{v}_a^{s}(k)u_a^{r}(p) \,, \tag{15.12}$$

where we can see a factor of $(-ig)$ at each vertex, the scalar propagator, and the external spinors for fermions and antifermions according to the rules derived earlier. Momentum

conservation fixes the value of the momentum flowing through the scalar propagator to $q = P + K = P' + K'$. To compute the squared of the amplitude we will need

$$-i\mathcal{A}^\dagger = (+ig)^2 \bar{u}_b^r(p) v_b^s(k) \, \frac{-i}{q^2 - \mu^2} \, \bar{v}_d^{s'}(k') u_d^{r'}(p') \ , \tag{15.13}$$

Thus, the amplitude squared is

$$|\mathcal{A}|^2 = \frac{g^4}{\left[q^2 - \mu^2\right]^2} \, u_a^r(p) \bar{u}_b^r(p) \, v_b^s(k) \bar{v}_a^s(k) \, v_c^{s'}(k') \bar{v}_d^{s'}(k') \, u_d^{r'}(p') \bar{u}_c^{r'}(p') \ . \tag{15.14}$$

If we now sum over all the initial and final spin indices, s, r s' and r', the we can make use of the identities

$$\begin{aligned} \sum_s u^s(p) \bar{u}^s(p) &= \not{p} + m \\ \sum_s v^s(p) \bar{v}^s(p) &= \not{p} - m \ , \end{aligned}$$

etc. Then, we can write the amplitude squared (15.14) as

$$\begin{aligned} |\mathcal{A}|^2 &= \frac{g^4}{\left[q^2 - \mu^2\right]^2} \, (\not{p} + m_\chi)_{ab} \, (\not{k} - m_\chi)_{ba} \, (\not{k}' - m_\psi)_{cd} \, (\not{p}' + m_\psi)_{dc} \ , \\ &= \frac{g^4}{\left[q^2 - \mu^2\right]^2} \, \mathrm{Tr}\left[(\not{p} + \mathrm{m}_\chi)(\not{k} - \mathrm{m}_\chi)\right] \, \mathrm{Tr}\left[(\not{k}' - \mathrm{m}_\psi)(\not{p}' + \mathrm{m}_\psi)\right] \ , \end{aligned} \tag{15.15}$$

where in the last equality we used the fact that the spinor indices in the first line are summed over in order to write the traces over them. The next step is to perform the traces over the gamma matrices. In this example this is very simple. For instance, we need to compute

$$\mathrm{Tr}\left[(\not{p} + \mathrm{m}_\chi)(\not{k} - \mathrm{m}_\chi)\right] = \mathrm{Tr}\left[\not{p}\,\not{k}\right] - \mathrm{m}_\chi^2 \mathrm{Tr}\left[\mathbf{1}\right] \ , \tag{15.16}$$

where we used the fact that the trace of a gamma matrix is zero, and in the last term we have the trace over the 4×4 identity, which is just 4. Finally, to compute the remaining trace we need to use the Clifford algebra obeyed by the gamma matrices

$$\mathrm{Tr}\left\{\gamma^\mu, \gamma^\nu\right\} = 4\,\mathrm{g}^{\mu\nu} \ , \tag{15.17}$$

so this results in

$$\mathrm{Tr}\left[\not{\mathrm{p}}\ \not{\mathrm{k}}\right] = \mathrm{p}_\mu\,\mathrm{k}_\nu\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4\,\mathrm{p}\cdot\mathrm{k} \ , \tag{15.18}$$

where to obtain the last result we used (15.17) together with the cyclic property of traces. Then, for (15.16) we obtain

$$\mathrm{Tr}\left[\left(\not{\mathrm{p}} + \mathrm{m}_\chi\right)\left(\not{\mathrm{k}} - \mathrm{m}_\chi\right)\right] = 4\,\left(\mathrm{p}\cdot\mathrm{k} - \mathrm{m}_\chi^2\right) \ . \tag{15.19}$$

With the obvious replacements, the same result is obtained for the second factor in (15.15). Then, the squared of the amplitude can be written as

$$|\mathcal{A}|^2 = \frac{16\,g^4}{\left[q^2-\mu^2\right]^2}\,\left(p\cdot k - m_\chi^2\right)\,\left(p'\cdot k' - m_\psi^2\right) \ . \tag{15.20}$$

15.4 Computing the Cross Section

At this point, we can choose a reference frame to express the four-momenta in (15.20) as well as in the phase space in order to compute the cross section for the scattering process. However, it is also useful to express everything in terms of Lorentz invariant variables whenever possible. For this purpose we define the Mandelstam variables for the scattering process generically depicted in Figure 15.2

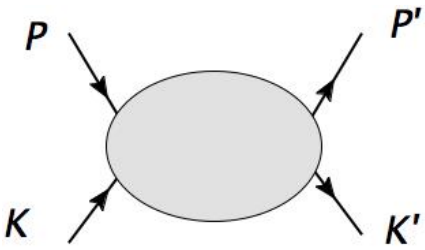

Figure 15.2: Generic kinematics of a $2 \to 2$ scattering used to defined the Mandelstam variables.

We can define three independent combinations of the four-momenta entering in the process once overall momentum conservation is imposed. With them we choose to form the following Lorentz invariants

$$\begin{aligned} s &\equiv (p+k)^2 = (p'+k')^2 \\ t &\equiv (p'-p)^2 = (k-k')^2 \\ u &\equiv (p'-k)^2 = (p-k')^2 \ . \end{aligned} \tag{15.21}$$

We can now express the momentum products in (15.20) in terms of these variables. Just to simplify the calculation a bit more, we will assume

$$m_\chi = 0, \qquad\qquad m_\psi = m \ , \tag{15.22}$$

from now on. Then, for the process $\chi\bar{\chi} \to \psi\bar{\psi}$, we have

$$s = (p+k)^2 = p^2 + k^2 + 2\,p\cdot k \ , \tag{15.23}$$

resulting in

$$\boxed{p\cdot k = \frac{s}{2}} \ . \tag{15.24}$$

Similarly

$$s = (p'+k')^2 = p'^2 + k'^2 + 2\,p'\cdot k' = 2(m^2 + p'\cdot k') \ , \tag{15.25}$$

resulting in

$$\boxed{p'\cdot k' = \frac{s-2m^2}{2}} \ . \tag{15.26}$$

Finally, we notice that by momentum conservation we have

$$\boxed{q^2 = (p+k)^2 = s} \ , \tag{15.27}$$

so the scalar propagator depends on s. When this is the case for a given Feynman diagram contribution, this is called "s channel". From (15.24), (15.26) and (15.27) we see that the amplitude squared depends exclusively on the Mandelstam variable s. Thus we can rewrite (15.20) as

$$\boxed{|\mathcal{A}|^2 = \frac{4\,g^4\,s^2\,\left(1-4m^2/s\right)}{\left[s-\mu^2\right]^2}}\,, \tag{15.28}$$

However, the amplitude squared in (15.28) was obtained summing over all the final *and* initial spin states. While this may be correct regarding the final spin states because the experiment is not measuring the individual spins of the final state (a typical situation), this is not correct regarding the initial states. This is because the scattering between the two initial states, $\chi\bar{\chi}$, takes place in one of all the possible spin combinations.

Figure 15.3: Possible spin configurations of unpolarized initial states in $\chi\bar{\chi}$ scattering.

This is illustrated in Figure 15.3, where we have four equally probable initial spin states corresponding to unpolarized beams. Then, the amplitude squared must reflect the average over these initial spin states we have "erroneously" summed over. In our case this implies we must use

$$\overline{|\mathcal{A}|^2} = \frac{1}{4}\,|\mathcal{A}|^2\ . \tag{15.29}$$

The denominator in (15.29) is the number of initial states that we might have summed over in the computation of the amplitude squared. In other cases, there might be additional quantum numbers we summed over in the calculation, such as color for initial quarks and/or gluons in quantum chromodynamics. These will result in additional factors in the denominator in (15.29). On the other hand, going back to our case of spins if the initial states are polarized it would be advantageous to not sum over the initial spin states and to work with separate amplitudes for each spin combination. These are called helicity amplitudes.

Finally, to compute the cross section we need to make use of

$$d\sigma = \frac{1}{2E_p\,2E_k}\,\frac{1}{|v_p - v_k|}\,d\Pi_2\,\overline{|\mathcal{A}|^2}\ , \tag{15.30}$$

where $v_{p,k}$ are the velocities of the initial fermion and antifermion, and the two-particle final-state phase space in the center of momentum (CM) frame is

$$d\Pi_2 = d\Omega \, \frac{1}{16\pi^2 E_{\rm CM}} \, |\mathbf{p}'| \ , \tag{15.31}$$

as previously derived. Here, $\mathbf{p}'$ is the spatial momentum of the final state particles. The differential solid angle is

$$d\Omega = d\phi \, d\cos\theta \ , \tag{15.32}$$

where ϕ is the azimuthal angle of the final state fermion, and θ is the angle between the direction of the incoming fermion χ and the one of the outgoing fermion ψ, as shown in Figure 15.4.

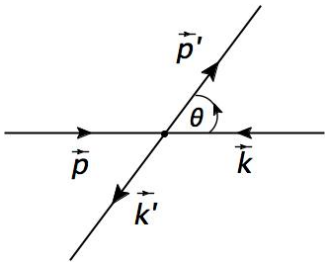

Figure 15.4: Kinematics of the $2 \to 2$ scattering in the CM frame. In particular $\mathbf{k} = -\mathbf{p}$, and $\mathbf{k}' = -\mathbf{p}'$.

In the CM frame we have that

$$E_{\rm CM} = E_p + E_k = E_{p'} + E_{k'} \ , \tag{15.33}$$

Then we can write

$$|\mathbf{p}'| = \sqrt{E_{p'}^2 - m^2} = \sqrt{E_{\rm CM}/4 - m^2} = \frac{E_{\rm CM}}{2} \sqrt{1 - 4m^2/E_{\rm CM}^2} \ , \tag{15.34}$$

where we used that $E_{\rm CM} = 2\, E_{p'}$.
Also in the CM frame, the denominator in the expression (15.30) for the cross section can be written as

$$2E_p \, 2E_k \, |v_p - v_k| = E_{\rm CM}^2 \, |1 - (-1)| = 2\, E_{\rm CM}^2 \ , \tag{15.35}$$

where we used $m_\chi = 0$ to obtain $v_p = 1$ and $v_k = -1$, and we used that $\mathbf{p} = |\mathbf{p}| \, \hat{z}$. Putting it all together, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\sqrt{1-4m^2/s}}{64\pi^2 s}\,\frac{g^4\,s^2\,(1-4m^2/s)}{\left[s-\mu^2\right]^2}\ , \tag{15.36}$$

which we can simplify to

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\,\frac{g^4\,s\,\left(1-4m^2/s\right)^{3/2}}{\left[s-\mu^2\right]^2}\ . \tag{15.37}$$

Let us make some comments on this result:

1. First we see that there is a threshold factor given by

$$\sqrt{1-\frac{4m^2}{s}}\ , \tag{15.38}$$

which clearly tells us that for the cross section to be non-zero we need to satisfy that $s \geq 4m^2$, i.e. the CM energy has to satisfy $E_{\rm CM} \geq 2\,m$. This threshold factor (15.38) comes directly from the expression (15.34) for the final state momentum and therefore is purely kinematic and is generically present independently of the details of the interactions. However, as it can be seen in (15.36), there are two additional powers of this threshold factor coming from the amplitude squared. So in the particular case of Yukawa theory, fermion–antifermion scattering (or fermion–antifermion annihilation) is suppressed by *three* powers of the threshold factor (15.38). This extra powers are not present in the case of other interactions, such as for instance for photon exchange in quantum electrodynamics (QED). In this way, we see that the scattering cross section behavior just above threshold is very different in Yukawa theory and in QED, even if the initial and final states were the same.

2. For CM energies much larger than both masses, i.e. for $s \gg m^2, \mu^2$, the cross section behaves like

$$\frac{d\sigma}{d\Omega} \sim \frac{1}{s}\ , \tag{15.39}$$

which is a generic behavior. Cross sections fall off with the squared of the CM energy for large enough energies, well above all mas scales in the problem.

3. For very large masses for the intermediate scalar, but CM energies still large when compared with m, i.e. for

$$m^2 \ll s \ll \mu^2\ , \tag{15.40}$$

we observe that the cross section behaves like

$$\frac{d\sigma}{d\Omega} \sim \frac{g^4}{64\pi^2}\,\frac{s}{\mu^2}, \tag{15.41}$$

that is, *it growths* with s. Having $\mu^2 \gg s, m^2$ suggests that we can obtain this results by integrating out the scalar to begin with. In fact this corresponds to the replacement of the scalar propagator by

$$\frac{i}{s-\mu^2} \longrightarrow \frac{-i}{\mu^2}\ . \tag{15.42}$$

Alternatively, we can obtain the result of (15.41) by integrating out ϕ from the lagrangian (15.1) using functional integral techniques. This would result in the following four-fermion interaction lagrangian

$$\mathcal{L}_{4f} = -\frac{g^2}{\mu^2}\,\bar{\chi}\chi\,\bar{\psi}\psi\ , \tag{15.43}$$

which is a dimension-six operator. Had we started with (15.43) we would know that since the coupling must have two negative powers of energy as units this is non-renormalizable interaction. The behavior of the cross section with s tells us that non-renormalizable interactions will result in a growing cross section, which in turn could result in a violation of unitarity (probability cannot grow indefinitely or else it could go over 100 % !) for a given value of the center of mass energy squared, s . This tells use that when we have a non-renormalizable interaction, the energy scale defined by its coupling (in our case μ) cannot be arbitrarily high or else unitarity will be violated.