Lecture 14

Feynman Rules for Fermions

We have already derived Feynman rules for scalar theories in perturbation theory. The next step is to do the same for a theory involving fermions. The simplest such theory is Yukawa theory, where fermions are coupled to a real scalar field through a dimension-four operator, resulting in a dimensionless coupling. But before we derive the Feynman rules for computing amplitudes in this theory, we need to know how to obtain amplitudes involving fermions from the appropriate correlation functions, i.e. we need to derive the LSZ formula for fermions.

14.1 LSZ Formalism for Fermions

The derivation of the LSZ formula for fermions folllows very closely the one for scalar fields. First, we define momentum states with the following normalizations

$$\begin{aligned} |p,s\rangle &= \sqrt{2E_p}\, a_p^{s\dagger}\,|0\rangle \\ |\overline{p,s}\rangle &= \sqrt{2E_p}\, b_p^{s\dagger}\,|0\rangle \quad , \end{aligned} \tag{14.1}$$

where s denotes the spin index, and the second state refers to an antifermion of momentum p and spin s. With the expansion of the fermion field given by

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_k}}\sum_s \left\{ a_k^s\, e^{-ik\cdot x}\, u^s(k) + b_k^{s\dagger}\, e^{+ik\cdot x}\, v^s(k)\right\} \ , \tag{14.2}$$

we can write the creation operators in terms of the field and the spinors as

$$
\begin{aligned}
a_p^{s\dagger} &= \frac{1}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x}\, \bar{\psi}(x)\gamma^0 u^s(p) \\
b_p^{s\dagger} &= \frac{1}{\sqrt{2E_p}} \int d^3x e^{-ip\cdot x}\, \bar{v}^s(p)\gamma^0 \psi(x)
\end{aligned} \tag{14.3}
$$

where we have used the normalization of spinors as

$$
u^{s\dagger}(p)u^r(p) = 2E_p\,\delta^{sr}, \qquad v^{s\dagger}(p)v^r(p) = 2E_p\,\delta^{sr}\ . \tag{14.4}
$$

We are interested in obtaining the transition amplitude between initial and final states, each of which is composed of asymtotic particle states. Just as we did in the case of scalar fields, we consider the asymtotic states in the far past making up the initial state $|i\rangle$ as created by creation operators evaluated at $t \to -\infty$. For instance, if the initial state consists of two fermions of momenta p_1 and p_2 we have

$$
|i\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a_{p_1}^{s_1\dagger}(-\infty)\, a_{p_2}^{s_2\dagger}(-\infty)\, |0\rangle\ , \tag{14.5}
$$

and analogously for the final state $|f\rangle$ defined in the far future or $t \to +\infty$. If we consider the scattering of two fermions in the initial state going into two fermions in the final state, it would have an amplitude given by

$$
\langle f_3 f_4 | f_1 f_2\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\, \langle 0| a_{p_3}(+\infty) a_{p_4}(+\infty)\, a_{p_1}^{\dagger}(-\infty)\, a_{p_2}^{\dagger}(-\infty)\, |0\rangle\ , \tag{14.6}
$$

where we have ommited the spinor indices for simplicity. In analogy with the scalar field case we have

$$
\begin{aligned}
\sqrt{2E_p}\,\left(a_p^\dagger(+\infty) - a_p^\dagger(-\infty)\right) &= \sqrt{2E_p}\int dt\,\partial_0\left(a_p^\dagger(t)\right) \\
&= \int d^4x\,\bar{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} - ip_0\gamma_0\right)\, u^s(p)\, e^{-ip\cdot x} \\
&= \int d^4x\,\bar{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} - ip_j\gamma^j - im\right)\, u^s(p)\, e^{-ip\cdot x} \\
&= \int d^4x\,\bar{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} - \gamma^j\overrightarrow{\partial_j} - im\right)\, u^s(p)\, e^{-ip\cdot x} \\
&= \int d^4x\,\bar{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} + \gamma^j\overleftarrow{\partial_j} - im\right)\, u^s(p)\, e^{-ip\cdot x}
\end{aligned} \tag{14.7}
$$

where $\overleftarrow{\partial}$ means that the derivative acts on the left of the operator. In the first line we use just basic calculus. In the second line we applied the time derivative on the first of equations (14.3). In the third line, we used that $(\not{p} - m)u^s(p) = 0$, and $j = 1, 2, 3$ are the spatial components. In the fourth line we use that $p_j = -i\partial_j$, and in the fifth one we integrate by parts the term with the spatial derivatives so that it acts on the left now. The final expression reads

$$\boxed{\sqrt{2E_p}\,\left(a_p^\dagger(+\infty) - a_p^\dagger(-\infty)\right) = -i\int d^4x\,\bar{\psi}(x)\left(i\overleftarrow{\not{\partial}}_x + m\right)u^s(p)\,e^{-ip\cdot x}}\,. \tag{14.8}$$

It is now straightforward to obtain the analogous expression for the annihilation operators, just by taking the complex conjugate operation on (14.8). This results in

$$\boxed{\sqrt{2E_p}\,\left(a_p(+\infty) - a_p(-\infty)\right) = -i\int d^4x\,e^{ip\cdot x}\,\bar{u}^s(p)\,(i\,\not{\partial}_x - m)\,\psi(x)}\,. \tag{14.9}$$

We can then use these expressions to write the LSZ formula for the fermion-fermion scattering process, $\langle f_3 f_4 | f_1 f_2\rangle$. For this process we can write the LSZ formula like

$$\begin{aligned}
\langle f_3 f_4 | f_1 f_2\rangle &= \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\,\langle 0|T a_{p_3}(+\infty)a_{p_4}(+\infty)\,a_{p_1}^\dagger(-\infty)\,a_{p_2}^\dagger(-\infty)\,|0\rangle \\
&= (-i)^4\int d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,e^{ip_3\cdot x_3}\,\bar{u}(p_3)\,(i\,\not{\partial}_{x_3} - m)\;e^{ip_4\cdot x_4}\,\bar{u}(p_4)\,(i\,\not{\partial}_{x_4} - m) \\
&\quad \langle 0|T\left(\psi(x_3)\psi(x_4)\bar{\psi}(x_1)\bar{\psi}(x_2)\right)|0\rangle\left(i\overleftarrow{\not{\partial}}_{x_1} + m\right)u(p_1)\,e^{-ip_1\cdot x_1}\;\left(i\overleftarrow{\not{\partial}}_{x_2} + m\right)u(p_2)\,e^{-ip_2\cdot x_2}
\end{aligned} \tag{14.10}$$

As we did for the scalar fields, we use the fact that the annihilation and creation operators are time ordered to express them as the time-ordered differences in (14.7) and (14.8) in the first equality above. The correlation function, i.e. the vacuum expectation value of the time-ordered product of the fermion fields in the external points, reflects the fact that in order to create fermions in the initial state we need $\bar{\psi}(x)$, whereas to create fermions in the final state $\psi(x)$ is needed. The salient point is that, analogously to the case of scalar fields, the action of the Dirac operators on the correlation function expressed as the sum of products of propagators, will result in delta functions that will organize the momentum conservation by having momentum delta functions at each vertex, in addition to stripping the amplitude of the external propagators.

Similarly, if we want the amplitude for scattering of a fermion and antifermion in the initial state going into a fermion and antifermion final state we will have

$$\langle f_3\bar{f}_4|f_1\bar{f}_2\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\,\langle 0|a_{p_3}(+\infty)b_{p_4}(+\infty)\,a^\dagger_{p_1}(-\infty)\,b^\dagger_{p_2}(-\infty)\,|0\rangle\ . \tag{14.11}$$

In the same way we obtained (14.7), we can obtain

$$\begin{aligned}\sqrt{2E_p}\,\left(b^\dagger_p(+\infty)-b^\dagger_p(-\infty)\right) &= \sqrt{2E_p}\int dt\,\partial_0\left(b^\dagger_p(t)\right)\\ &= -i\int d^4x\,e^{-ip\cdot x}\,\bar{v}^s(p)\,(i\,\partial\!\!\!/_x-m)\,\psi(x)\ ,\end{aligned}$$

for the anti-fermion creation operators. This is

$$\boxed{\sqrt{2E_p}\,\left(b^\dagger_p(+\infty)-b^\dagger_p(-\infty)\right) = -i\int d^4x\,e^{-ip\cdot x}\,\bar{v}^s(p)\,(i\,\partial\!\!\!/_x-m)\,\psi(x)}\ . \tag{14.12}$$

For the annihilation operators we simply obtain, which by complex conjugation,

$$\boxed{\sqrt{2E_p}\,\left(b_p(+\infty)-b_p(-\infty)\right) = -i\int d^4x\,\bar{\psi}(x)\left(i\overleftarrow{\partial\!\!\!/}_x+m\right)v^s(p)\,e^{ip\cdot x}}\ , \tag{14.13}$$

which results in the corresponding LSZ formula

$$\begin{aligned}\langle f_3\bar{f}_4|f_1\bar{f}_2\rangle &= \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\,\langle 0|T\,a_{p_3}(+\infty)b_{p_4}(+\infty)\,a^\dagger_{p_1}(-\infty)\,b^\dagger_{p_2}(-\infty)\,|0\rangle\\ &= (-i)^4\int d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,e^{ip_3\cdot x_3}\,\bar{u}(p_3)\,(i\,\partial\!\!\!/_{x_3}-m)\;e^{ip_4\cdot x_4}\,v(p_4)\,(i\,\partial\!\!\!/_{x_4}+m)\\ &\quad\times\langle 0|T\left(\psi(x_3)\bar{\psi}(x_4)\bar{\psi}(x_1)\psi(x_2)\right)|0\rangle\\ &\quad\times\left(i\overleftarrow{\partial\!\!\!/}_{x_1}+m\right)u(p_1)\,e^{-ip_1\cdot x_1}\;\left(i\overleftarrow{\partial\!\!\!/}_{x_2}-m\right)\bar{v}(p_2)\,e^{-ip_2\cdot x_2}\end{aligned} \tag{14.14}$$

More generally, we can use the boxed equations (14.8), (14.9), (14.12) and (14.13) to compute any amplitude for any process involving fermions and/or anti-fermions in the initial and/or final state.

14.2 Yukawa Theory

Having derived the LSZ formula for fermions we are now in a position to derive the Feynman rules in a theory coupling a fermion with a real scalar, Yukawa theory[1]. The lagrangian density is given by

$$\mathcal{L} = \bar{\psi}\left(i\,\partial\!\!\!/ + m\right)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\bar{\psi}\psi\phi + \dots\ , \tag{14.15}$$

where the dots include several terms that must be present because they will be generated by fermion loop corrections, such as *tadpole* term ϕ, and ϕ^3 and ϕ^4 terms. We are however only interested in the triple interaction fermion-fermion-scalar above, since we will only work at tree level in this lecture[2].

The first step to derive the Feynman rules is to obtain the rule for the vertex at tree level. For this purpose, as well as to obtain the subsequent correlation functions, we need the generating functional

$$Z[\eta,\bar{\eta},J] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS[\psi,\bar{\psi},\phi]+i\int d^4x\,\left\{J(x)\phi(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right\}} \tag{14.16}$$

The vertex at tree level is obtained as a 3-point function given by

$$G^{(3)}(x_1,x_2,x_3) = \frac{(-i)^3}{Z[0,0,0]}\ \frac{\delta^3 Z[\eta,\bar{\eta},J]}{\delta J(x_1)\delta\eta(x_2)\delta\bar{\eta}(x_3)}\bigg|_{J,\eta,\bar{\eta}=0} \tag{14.17}$$

In order to implement perturbation theory in powers of the coupling constant g we expand the interaction term in the action. To obtain the desired correlation function we only need to go up to the $O(g)$ term

$$\begin{aligned} G^{(3)}(x_1,x_2,x_3) = \frac{1}{Z[0,0,0]}\int &\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS_0[\psi,\bar{\psi},\phi]}\phi(x_1)\bar{\psi}(x_2)\psi(x_3) \\ &\times\left(1 - ig\int d^4y\,\bar{\psi}(y)\psi(y)\phi(y)\ + \dots\right)\ , \end{aligned} \tag{14.18}$$

where the dots denote terms of order g^2 and higher and S_0 refers to the action of the free theory, or the order g^0 action. Using Wick's theorem we see that there is only one

[1] We use this name for short. However, true Yukawa theory requires ϕ to be a pseudo-scalar, i.e. odd under parity transformations.

[2] In true Yukawa theory parity invariance forbids the presence of odd powers of ϕ. However, the ϕ^4 term is still inevitably generated at one loop.

possible pairing into propagators. To get the sign right, we must remember that fermion fields anticommute. Thus, the pairing of fields into propagators gives

$$\begin{aligned}\phi(x_1)\bar{\psi}(x_2)\psi(x_3)\bar{\psi}(y)\psi(y)\phi(y) &= D_F(x_1-y)\,\bar{\psi}(x_2)\,S_F(x_3-y)\,\psi(y) \\ &= -\,D_F(x_1-y)\,S_F(x_3-y)\,\psi(y)\bar{\psi}(x_2) \\ &= -\,D_F(x_1-y)\,S_F(x_3-y)\,S_F(x_2-y)\;. \quad (14.19)\end{aligned}$$

Then, the use of Wick's theorem results in

$$G^{(3)}(x_1,x_2,x_3) = +ig\int d^4y\,D_F(x_1-y)\,S_F(x_3-y)_{ab}\,S_F(x_2-y)_{ba}\;, \qquad (14.20)$$

where a and b are Dirac indices and are summed over. In position space we represent this by Figure 14.1.

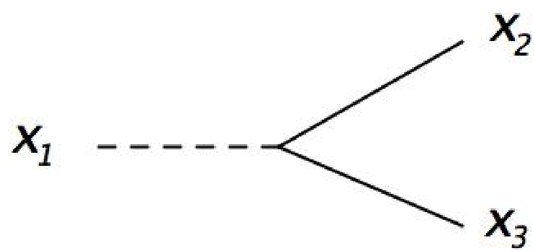

Figure 14.1: Position space Feynman diagram for the 3-point function in Yukawa theory. The dashed line representes the scalar propagator, whereas solid lines stand for the fermion propagators.

To obtain the Feynman rule in momentum space we make use of the LSZ formula. Since the amplitude is mixed, having fermion and scalar external propagators, the LSZ reduction formula involves Klein-Gordon operators as well as Dirac operators. We start with the expression in terms of annihilation and creation operators

$$\langle f_2\,\bar{f}_3|s_1\rangle = \sqrt{2E_1}\,\sqrt{2E_2}\,\sqrt{2E_3}\,\langle 0|a_{p_2}(+\infty)\,b_{p_3}(+\infty)\,c^\dagger_{p_1}(-\infty)|0\rangle\;, \qquad (14.21)$$

where we defined $c^\dagger_{p_1}(t)$ as the creation operator for a scalar of momentum p_1 at time t. Using (14.9) and (14.13), as well as a similar expression for the creation operator of the scalar field, we have

$$\begin{aligned}\langle f_2\,\bar{f}_3|s_1\rangle &= (-i)^3\int d^4x_1\,d^4x_2\,d^4x_3\,e^{-ip_2\cdot x_2}\,\bar{u}^s(p_2)\,(i\,\partial\!\!\!/_{x_2}-m)\;e^{-ip_1\cdot x_1}\left(\partial^2_{x_1}+\mu^2\right)\times\\ &\qquad\langle 0|T\,\phi(x_1)\,\psi(x_2)\,\bar{\psi}(x_3)|0\rangle\left(i\overleftarrow{\partial\!\!\!/}_{x_3}+m\right)v^r(p_3)\,e^{ip_3\cdot x_3}\;, \end{aligned}\tag{14.22}$$

If we now use the expression (14.20) in (14.22), and using that

$$(\partial^2_{x_1}+\mu^2)\,D_F(x_1-y) = -i\delta^{(4)}(x_1-y)\;,\tag{14.23}$$

and that

$$(i\,\partial\!\!\!/_{x_2}-m)\,S_F(x_2-y) = i\delta^{(4)}(x_2-y),\qquad S_F(x_3-y)(i\overleftarrow{\partial\!\!\!/}_{x_3}+m) = i\delta^{(4)}(x_3-y)\;,\tag{14.24}$$

we arrive at

$$\langle f_2\,\bar{f}_3|s_1\rangle = (-ig)\,(2\pi)^4\,\delta^{(4)}(P_1-P_2-P_3)\,\bar{u}^s(p_2)\,v^r(p_3)\;,\tag{14.25}$$

where we used the delta functions to perform the integrals over the external points in (14.22) and the integral over the internal point y in the correlation function results in the momentum conservation delta function. Here we considered the fermion and antifermion as outgoing, with the incoming scalar[3]. Thus, the momentum space Feynman rule for the vertex in Yukawa theory tells us to have a factor of $-ig$ at the vertex, a momentum conservation delta function, plus a factor of $\bar{u}(p)$ for each outgoing fermion, and $v^s(p)$ for each outgoing antifermion. This defines the vertex for the Yukawa theory. In general, for each occurring vertex we will need to have a factor of $(-ig)$, regardless of the fact that the fermions are external asymptotic states or there are internal to a diagram and therefore represented by their propagators. The spinors will only be present if the fermions are indeed external states.

A more interesting process to consider is that of fermion–antifermion scattering. We consider $f_1\bar{f}_2\to f_3\bar{f}_4$. The relevant correlation function is

$$G^{(4)}(x_1,x_2,x_3,x_4) = \frac{(-i)^4}{Z[0,0,0]}\,\frac{\delta^4}{\delta\eta(x_4)\delta\bar{\eta}(x_3)\delta\bar{\eta}(x_2)\delta\eta(x_1)}Z[J,\eta,\bar{\eta}]\Big|_{J,\eta,\bar{\eta}=0}\;,\tag{14.26}$$

where once again we have chosen the functional derivatives applied on (14.16) so as to obtain the desired mix of fermions and antifermions in the correlation function. The first

[3] This is of course arbitrary and unnecessary. When the vertex appears in a given diagram, the approrpiate spinors should be inserted. The actual content of the vertex rule is just the factor $(-ig)$ and the momentum conservation delta function.

non-zero contribution to this process comes from the second order term in the expansion of the interaction term in the action.

$$\begin{aligned} G^{(4)}(x_1,x_2,x_3,x_4) &= \frac{1}{Z[0,0,0]}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\, e^{iS_0[\psi\bar{\psi},\phi]}\,\bar{\psi}(x_4)\psi(x_3)\psi(x_2)\bar{\psi}(x_1) \\ &\quad\times\left(1-ig\int d^4y\,\bar{\psi}(y)\psi(y)\phi(y)\right. \\ &\quad\left.+\frac{(-ig)^2}{2!}\int d^4y\,d^4z\,\bar{\psi}(y)\psi(y)\phi(y)\,\bar{\psi}(z)\psi(z)\phi(z)+\dots\right)\,. \end{aligned} \tag{14.27}$$

Taking all possible pairings according to Wick's theorem, we see there are two such possible terms. We must be careful in making the pairings in the process of contracting fields of different points into fermion propagators due to the anticommutation rules for spinor fields. For the first pairing we have

$$\begin{aligned} \bar{\psi}(x_4)\psi(x_3)\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(y)\psi(y)\bar{\psi}(z)\psi(z) &= +\bar{\psi}(x_4)\psi(x_3)\psi(x_2)\bar{\psi}(y)S_F(x_1-y)\bar{\psi}(z)\psi(z) \\ &= +\bar{\psi}(x_4)\psi(x_3)S_F(x_2-y)S_F(x_1-y)\bar{\psi}(z)\psi(z) \\ &= +\bar{\psi}(x_4)S_F(x_3-z)S_F(x_2-y)S_F(x_1-y)\psi(z) \\ &= -S_F(x_4-z)S_F(x_3-z)S_F(x_2-y)S_F(x_1-y)\,. \end{aligned} \tag{14.28}$$

To understand the sign in the last equation in (14.28), we first see that in the first line we need to pass $\bar{\psi}(x_1)$ through two fermion fields, $\bar{\psi}(y)$ and $\psi(y)$, in order to form $S_F(x_1-y)$ with the latter. Then, the second line is obtained by noticing that $S_F(x_2-y)$ is obtained from the previous line without moving any fermion field. For the third line, we move freely (without anticommuting) $\bar{\psi}(z)$ to the left through the two propagators in order to form $S_F(x_3-z)$. Finally, to obtain the last line, we need to flip $\bar{\psi}(x_4)$ with $\psi(z)$ in order to form $S_F(x_4-z)$, which picks up a minus sign due to the anticommutation rules. Thus, the first contribution has an overall negative sign.

We repeat the procedure for the second possible contractions. We now have

$$\begin{aligned} \bar{\psi}(x_4)\psi(x_3)\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(y)\psi(y)\bar{\psi}(z)\psi(z) &= +\bar{\psi}(x_4)\psi(x_3)\psi(x_2)\bar{\psi}(y)S_F(x_1-y)\bar{\psi}(z)\psi(z) \\ &= -\bar{\psi}(x_4)\psi(x_2)S_F(x_3-y)S_F(x_1-y)\bar{\psi}(z)\psi(z) \\ &= -\bar{\psi}(x_4)S_F(x_3-y)S_F(x_1-y)S_F(x_2-z)\psi(z) \\ &= +S_F(x_4-z)S_F(x_3-y)S_F(x_1-y)S_F(x_2-z)\,. \end{aligned} \tag{14.29}$$

Then, the correlation function is given by

$$G^{(4)}(x_1,x_2,x_3,x_4) = (-ig)^2 \int d^4y\, d^4z\, \{-S_F(x_1-y)S_F(x_2-y)D_F(y-z)S_F(x_3-z)S_F(x_4-z) + S_F(x_1-y)S_F(x_2-z)D_F(y-z)S_F(x_3-y)S_F(x_4-z)\}\ , \tag{14.30}$$

meaning that the two contributions have a *relative* sign between them. We can represent these two contributions by Feynman diagrams in position space

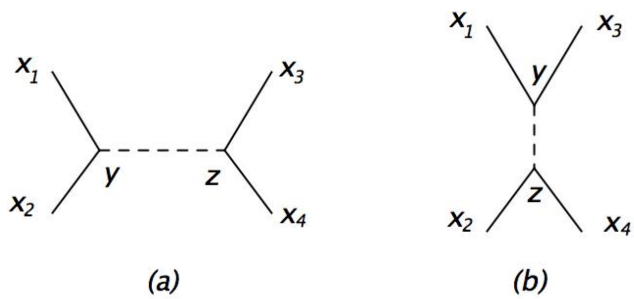

Figure 14.2: Connected position-space diagrams contributing to the four-point function to order g^2 in Yukawa theory. Solid lines denote fermion propagators, whereas dashed lines correspond to scalar ones.

If we now use the LSZ formula to obtain the amplitude for the fermion–antifermion scattering process, the external propagators will be stripped by the action of the Dirac operators on them, leaving delta functions. The result is, for the contribution of the diagram from Figure 14.2 (a)

$$\begin{aligned}
\langle f_3\bar{f}_4|f_1\bar{f}_2\rangle_{(a)} &= -(-ig)^2\int d^4y\, d^4z \int d^4x_1\, d^{-iP_1\cdot x_1}\delta(x_1-y)\int d^4x_2\, e^{-iP_2\cdot x_2}\delta(x_2-y)D_F(y-z)\\
&\quad \int d^4x_3 e^{iP_3\cdot x_3}\delta(x_3-z)\int d^4x_4 e^{iP_4\cdot x_4}\delta(x_4-z)\,\bar{v}^{s_2}(p_2)u^{s_1}(p_1)\,\bar{u}^{s_3}(p_3)v^{s_4}(p_4)\\
&= -(-ig)^2\int d^4y\, d^4z\, e^{-i(P_1+P_2)\cdot y}\, D_F(y-z)\, e^{i(P_3+P_4)\cdot z}\,\bar{v}^{s_2}(p_2)\ldots v^{s_4}(p_4)\\
&= -(-ig)^2\int d^4y\, d^4z\int \frac{d^4q}{(2\pi)^4}e^{-iq\cdot(y-z)}\frac{i}{q^2-\mu^2}e^{-i(P_1+P_2)\cdot y}\, e^{i(P_3+P_4)\cdot z}\\
&\quad \times\bar{v}^{s_2}(p_2)u^{s_1}(p_1)\,\bar{u}^{s_3}(p_3)v^{s_4}(p_4)\ ,
\end{aligned} \tag{14.31}$$

where in the last equality we expressed the scalar propagator $D_F(y-z)$ by its momentum integral. It is clear from (14.31) that we can now perform the integrals on y and z obtaining

$$
\begin{aligned}
\langle f_3\bar{f}_4|f_1\bar{f}_2\rangle_{(a)} &= -(-ig)^2\int\frac{d^4q}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(P_1+P_2+q)\,(2\pi)^4\delta^{(4)}(P_3+P_4+q) \\
&\quad \frac{i}{q^2-\mu^2}\,\bar{v}^{s_2}(p_2)u^{s_1}(p_1)\,\bar{u}^{s_3}(p_3)v^{s_4}(p_4) \\
&= -(-ig)^2(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4)\,\frac{i}{q^2-\mu^2}\,\bar{v}^{s_2}(p_2)u^{s_1}(p_1)\,\bar{u}^{s_3}(p_3)v^{s_4}(p_4)\ ,
\end{aligned}
\tag{14.32}
$$

where by virtue of the last q integration $q=-(P_1+P_2)=-(P_3-P_4)$ in the scalar propagator. The expression in (14.32) is the contribution to the amplitude of the diagram Figure 14.2 (a), but in momentum space. This is depicted by the momentum-space Feynman diagram of Figure 14.3 (a).

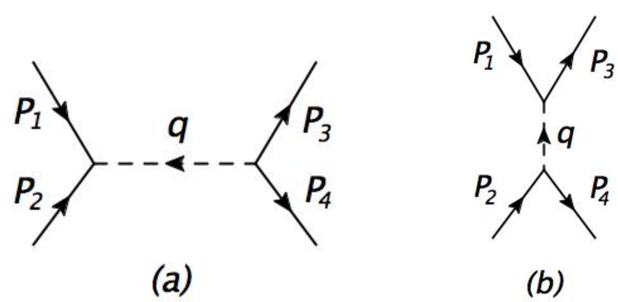

Figure 14.3: Connected momentum-space diagrams contributing to the scattering amplitude up to order g^2 in Yukawa theory. Solid lines denote fermion propagators, whereas dashed lines correspond to scalar ones.

By a completely analogous procedure we obtain the contribution to the amplitude from the diagram in Figure 14.2 (b). The answer is

$$
\langle f_3\bar{f}_4|f_1\bar{f}_2\rangle_{(b)} = (-ig)^2(2\pi)^4\delta(P_1+P_2-P_3-P_4)\,\frac{i}{q^2-\mu^2}\,\bar{u}^{s_3}(p_3)v^{s_1}(p_1)\,\bar{v}^{s_2}(p_2)u^{s_4}(p_4)\ ,
\tag{14.33}
$$

where the integral over q is done with a delta function fixing it at $q=P_3-P_1=P_2-P_4$, i.e. different from the value of the momentum flowing through the scalar propagator the previous amplitude. This contribution to the amplitude is depicted in a momentum-space Feynman diagram in Figure 14.3 (b). The *relative* sign between the two amplitudes is an observable since when computing the square of the amplitude the sign of the interference term will depende on it.

From this exercise we can derive the Feynman rules for Yukawa theory. The diagrams of Figure 14.3 can be used to obtain the desired amplitude for the scattering process $f_1\bar{f}_2 \to f_3\bar{f}_4$ by following these rules.

1. Insert a factor of $(-ig)$ for each vertex in the diagram.
2. Insert a factor of
$$\frac{i}{q^2 - \mu^2}$$
for each scalar propagator of momentum q in the diagram.
3. Insert a factor of
$$\frac{i}{\not{p} - m}$$
for every fermion propagator of momentum p in the diagram.
4. Momentum conservation is satisfied at each vertex. There is an overall momentum conservation factor of
$$(2\pi)^4\,\delta^{(4)}(P_i - P_f)$$
in the contribution of each diagram to the total amplitude, where P_i is the sum of all the initial four-momenta, and P_f is the sum of all final four-momenta.
5. Integrate over undetermined four-momenta (if any) with a factor of
$$\int \frac{d^4k}{(2\pi)^4}$$
for each free four-momentum k in the diagram.
6. Divide by any symmetry factors.
7. Attach a factor of:
   - $u^s(p)$ for each incoming fermion of momentum p,
   - $\bar{u}^s(p)$ for each outgoing fermion,
   - $\bar{v}^s(p)$ for each incoming antifermion,
   - $v^s(p)$ for each outgoing antifermion.
8. Insert a factor of (-1) for each enclosed fermion loop.

These are the momentum-space Feynman rules for computing amplitudes of processes. One needs to draw all possible diagrams contributing, and also be careful to have the correct relative sign between the various diagrams, such as in the case of the relative minus sign between the two diagrams of Figure 14.3. The final rule of the list above will be derived below.

14.3 Closed Fermion Loops

The last of the Feynman rules in the previous section refers to diagrams with closed fermion loops. For instance, let us consider the diagram of Figure 14.4 in Yukawa theory.

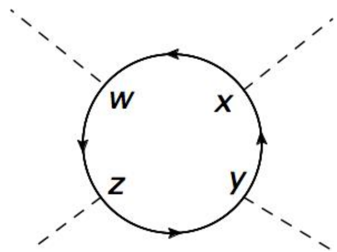

Figure 14.4: Closed fermion loop in Yukawa theory.

We can think of the diagram in position space as involving the product of fermions fields

$$\bar{\psi}_a(x)\psi_a(x)\,\bar{\psi}_b(y)\psi_b(y)\,\bar{\psi}_c(z)\psi_c(z)\,\bar{\psi}_d(w)\psi_d(w)\ , \tag{14.34}$$

where we have ignored the scalar fields since they do not enter in the proof, the internal positions are to be integrated over, and we made explicit the spinorial indexes. It is clear that in order to build the internal fermion propagators we need to get spinors to anticommute through an odd number of fermion fields. For instance, the entire diagram can be contracted in pair of fermions as indicated in Figure 14.4, if we get $\bar{\psi}_a(x)$ through all fields to contract with $\psi_d(y)$ and form $S_F(y-x)_{da}$. The answer would read

$$\begin{aligned} & (-1)^7 S_F(x-w)_{ab}\, S_F(w-z)_{bc}\, S_F(z-y)_{cd}\, S_F(y-x)_{da} \\ = \;& (-1)\,\mathrm{Tr}\left[\mathrm{S_F(x-w)\,S_F(w-z)\,S_F(z-y)\,S_F(y-x)}\right]\ . \end{aligned} \tag{14.35}$$

We can see from (14.35) that this is a generic feature of closed fermion loops: Since all fermion fields minus two are already arranged to contract in the appropriate propagators, we always need to move one fermion field through all the others to form the last propagator closing the loop. This always brings a factor of (-1) times the trace of the product of all propagators. That is, for a closed fermion loop with n_f fermion propagators, there is always factor of

$$(-1)^{2n_f-1}\,\mathrm{Tr}\left[\mathrm{S_F}\ldots\mathrm{S_F}\right] = (-1)\,\mathrm{Tr}\left[\prod_{\mathrm{i=1}}^{\mathrm{n_f}}\mathrm{S_{Fi}}\right]\ . \tag{14.36}$$

This is the last Feynman rule we need to compute amplitudes from Feynman diagrams in momentum space.

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 4.7.
- *Quantum Field Theory* , by M. Srednicki, Chapter 45.