Lecture 13

Cross Sections and Decay Rates

Now that we know how to compute amplitudes for given processes, we would like to make contact with observables such as cross sections and decay rates based on those amplitudes. This will complete the path from computing correlation functions and then amplitudes, which can be easily obtained by using the derived Feynman rules of a given theory.

13.1 The S Matrix

We will state the amplitude in the language of the S matrix. Let us consider a scattering process with a given initial state and a final state. We define the asymptotic states by

$$\begin{aligned} &|i, \mathrm{in}\rangle \qquad\qquad \text{for} \quad t \to -\infty \\ &|f, \mathrm{out}\rangle \qquad\qquad \text{for} \quad t \to +\infty \ , \end{aligned} \tag{13.1}$$

where the states labeled "in" are those asymptotic states created by creation operators evaluated at times $-\infty$, e.g. $a^\dagger(-\infty)$, etc; and the states labeled "out" are those created by creation operators evaluated at times $+\infty$, such as $a^\dagger(+\infty)$. These two distinct sets of asymptotic states are the ones we have used up until now to write down the desired amplitude

$$\langle f, \mathrm{out}|i, \mathrm{in}\rangle \ . \tag{13.2}$$

The "in" and "out" asymptotic states are however isomorphic, i.e. there are the same set of states but labeled differently. We can define a unitary transformation $\mathbf{S}$ such that

$$|i, \mathrm{in}\rangle = \mathbf{S}|i, \mathrm{out}\rangle \ , \tag{13.3}$$

in such a way that we can rewrite (13.2) in terms of either both “in” or“out” states.

$$\langle f,\mathrm{out}|i,\mathrm{in}\rangle = \langle f,\mathrm{in}|\mathbf{S}|i,\mathrm{in}\rangle = \langle f,\mathrm{out}|\mathbf{S}|i,\mathrm{out}\rangle \equiv \langle f|\mathbf{S}|i\rangle \ . \tag{13.4}$$

The last equality stems from the fact that we can equally express the amplitude in terms of the “in” or the “out” states as long as is an element of the **S** matrix. The **S** operator can be written as

$$\mathbf{S} \equiv \mathbf{1} + i\mathbf{T} \ , \tag{13.5}$$

where we defined the T matrix elements. The identity in the first term in (13.5) reflects the fact that the amplitude must include the possibility of no interaction. But in order to compute a cross section we are only concerned with the part of the amplitude that allows for interactions, i.e. the second term in (13.5). Schematically, we can express this as

$$\langle f|\mathbf{S}|i\rangle = \mathrm{disconnected\ \ diagrams} \qquad + \qquad \mathrm{LSZ\ \ formula}\ , \tag{13.6}$$

where the contributions of disconnected diagrams comes from the identity in (13.5). Thus, the LSZ formula will give the contribution of the T matrix to a given amplitude.

13.2 From Scattering Amplitudes to Cross Sections

In general we want to compute the transition probability from an initial state to a final state. In practice, we are mainly interested in two cases: the decay of a particle to two or more particles, and the scattering of two particles in the initial state into two or more particles in the final state.

We start with the scattering process $2 \to n$. The transition amplitude is given by

$$\langle \mathbf{p_1} \ldots \mathbf{p_n}|\, i\mathbf{T}\, |\mathbf{p_A} \mathbf{p_B}\rangle \equiv (2\pi)^4 \delta^{(4)}(P_A + P_B - P_1 - \cdots - P_n)\, i\,\mathcal{A} \ , \tag{13.7}$$

where we have defined the amplitude $\mathcal{A}$ as the transition amplitude with the overall momentum conservation delta function already factored out. In order to obtain a probability, we will define it as the squared of the transition amplitude appropriately normalized.

$$P \equiv \frac{|\langle \mathbf{p_1} \ldots \mathbf{p_n}|\, i\mathbf{T}\, |\mathbf{p_A} \mathbf{p_B}\rangle|^2}{\langle \mathbf{p_1} \cdots \mathbf{p_n}|\mathbf{p_1} \cdots \mathbf{p_n}\rangle \langle \mathbf{p_A} \mathbf{p_B}|\mathbf{p_A} \mathbf{p_B}\rangle} \ , \tag{13.8}$$

where the denominator corresponds to the normalization of the initial and final states.

We start by considering the numerator of (13.8). This is

$$\begin{aligned} |\langle \mathbf{p_1}\dots\mathbf{p_n}|\, i\mathbf{T}\,|\mathbf{p_A}\mathbf{p_B}\rangle|^2 &= \left((2\pi)^4\delta^{(4)}(P_A+P_B-\sum_{f=1}^{n}P_f)\right)^2 |\mathcal{A}|^2 \\ &= (2\pi)^4\delta^{(4)}(P_A+P_B-\sum_{f=1}^{n}P_f)\,(2\pi)^4\delta^{(4)}(0)\;|\mathcal{A}|^2\;, \end{aligned} \tag{13.9}$$

where $f=1,\dots,n$ labels the final state momenta. However, we can write

$$\delta^{(4)}(0)=\delta(0)\delta^{(3)}(0)=\frac{1}{(2\pi)^4}\int d^4x\, e^{i0\cdot x}\;. \tag{13.10}$$

If we consider for a moment a finite volume V and a finite time T, the integral in (13.10) results in

$$(2\pi)^4\delta^{(4)}(0)=VT\;. \tag{13.11}$$

For the denominator, we consider the asymptotic momentum eigenstates normalized according to

$$|\mathbf{p}\rangle=\sqrt{2E_p}\,a_p^\dagger|0\rangle\;, \tag{13.12}$$

such that the normalization of an eigenstate of momentum $\mathbf{p}$ is given by

$$\begin{aligned} \langle\mathbf{p}|\mathbf{p}\rangle &= 2E_p\,\langle 0|a_p\,a_p^\dagger|0\rangle \\ &= 2E_p\,(2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p})=2\,E_p\,V\;, \end{aligned} \tag{13.13}$$

where in the last equality we used (13.10). Then, the two factors in the denominator of (13.8) are

$$\begin{aligned} \langle\mathbf{p_A}\,\mathbf{p_B}|\mathbf{p_A}\,\mathbf{p_B}\rangle &= 2E_A\,2E_B\,V^2 \\ \langle\mathbf{p_1}\dots\mathbf{p_n}|\mathbf{p_1}\dots\mathbf{p_n}\rangle &= 2E_1\dots 2E_nV^n=\prod_f(2E_fV)\;. \end{aligned} \tag{13.14}$$

Replacing (13.10) and (13.14) into (13.8) and dividing by T, we obtain the probability of transition for unit time

$$\frac{P}{T} = \frac{(2\pi)^2\delta^{(4)}(P_A - P_B - \sum_f P_f)\, V\ |\mathcal{A}|^2}{2E_A\, 2E_B\, V^2\, \prod_f(2E_f V)}\ . \tag{13.15}$$

But this probability requires that we have precise knowledge of all final state momenta. Often times we will need to either partially or totally integrate over the phase space of the final states. For this we need to know the probability that a given final state particle has momentum in the interval

$$(\mathbf{p_f}, \mathbf{p_f} + d^3p_f)\ , \tag{13.16}$$

where d^3p_f contains information about the momentum vector. We would like then to convert (13.15) into the differential probability that the final states are in a region of the final state phase space defined by (13.16). In order to obtain this we need to multiply (13.15) by the number of states in each interval defined by (13.16) for each final state particle. Given that we are using a finite volume V, the momentum of each final state particle obeys the quantization rule

$$\mathbf{p} = \frac{2\pi}{L}\,(n_1, n_2, n_3)\ , \tag{13.17}$$

where $L^3 = V$, and the n_i with $i = 1, 2, 3$ refer to the number of states in each spatial direction. Then, the number of states inside the interval (13.16) of size d^3p is

$$\begin{aligned} n_1\, n_2\, n_3 &= \frac{Ldp_x}{2\pi}\frac{Ldp_y}{2\pi}\frac{Ldp_z}{2\pi} \\ &= \frac{V\, d^3p}{(2\pi)^3} \end{aligned} \tag{13.18}$$

Putting all these together we obtain the differential probability per unit time

$$\frac{dP}{T} = \frac{(2\pi)^2\,\delta^{(4)}(P_A + P_B - \sum_f P_f)\,|\mathcal{A}|^2}{2E_A\, 2E_B\, V} \prod_{f=1}^{n}\left(\frac{d^3p_f}{(2\pi)^3\, 2E_f}\right)\ , \tag{13.19}$$

Finally, in order to convert this into a differential cross section we need to account for the incident flux. In other words, we are interested in the differential probability per unit time *and* per unit of initial flux so that we obtain a probability that depends intrinsically on the amplitude $\mathcal{A}$ and the final state phase space, not on how intense our beams of A and B particles were. The flux is the number of particles per unit volume times the relative velocity of the particles. For instance, for a typical head on collision

Figure 13.1: Head on collision. $\mathbf{p_B} = -\mathbf{p_A}$.

the initial flux "seen" by either the A or the B particle is given by

$$\frac{|v_A^z - v_B^z|}{V} \ . \tag{13.20}$$

So dividing (13.19) by the flux in (13.20) we obtain

$$d\sigma = \frac{1}{2E_A\, 2E_B} \frac{1}{|\mathbf{v_A} - \mathbf{v_B}|} (2\pi)^4\, \delta^{(4)}(P_A + P_B - \sum_f P_f)\, |\mathcal{A}|^2 \prod_f \left(\frac{d^3 p_f}{(2\pi)^3\, 2E_f} \right) \ , \tag{13.21}$$

which is the differential cross section for the scattering of the two initial particles with momenta P_A and P_B going into an n-particle final state.
At this point we will make some comments:

- We can define the final state phase space by

$$\int d\Pi_n \equiv \int \prod_{f=1}^{n} \left(\frac{d^3 p_f}{(2\pi)^3\, 2E_f} \right) (2\pi)^4\, \delta^{(4)}(P_A + P_B - \sum_f P_f) \ . \tag{13.22}$$

  It is separately Lorentz invariant.

- The amplitude squared $|\mathcal{A}|^2$ is also Lorentz invariant by itself.

- The factor

$$\frac{1}{E_A\, E_B |v_A^z - v_B^z|} \ , \tag{13.23}$$

  is not Lorentz invariant, but it is invariant under boosts in the z direction.

13.3 Two-particle Final State

A very paradigmatic example is the scattering of two particles in the initial state into two particles in the final state. We first compute the two-particle phase space for $A+B \to 1+2$. We will use the center of momentum frame. From (13.22) we have

$$\begin{aligned}\int d\Pi_2 &= \int \frac{d^3p_1}{(2\pi)^3}\,\frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3}\,\frac{1}{2E_2}\,(2\pi)^4\,\delta^{(4)}(P_A+P_B-P_1-P_2)\\ &= \int \frac{d^3p_1}{(2\pi)^3}\,\frac{1}{4E_1E_2}\,2\pi\,\delta(E_A+E_B-E_1-E_2)\ ,\end{aligned} \tag{13.24}$$

where the second line is obtained by using the spatial delta function to perform the d^3p_2 integral. The final momentum differential is

$$d^3p_1 = p_1^2\,dp_1\,d\Omega_1 = p_1^2\,dp_1\,d\cos\theta_1\,d\phi_1\ , \tag{13.25}$$

with θ_1 the angle of $\mathbf{p_1}$ with respect to the direction of the incoming momentum $\mathbf{p_A}$, and ϕ_1 the corresponding azimuthal angle. There is typically no azimuthal angle dependence in $|\mathcal{A}|^2$, so we can integrate over ϕ_1 obtaining a factor of 2π. Then (13.24) now reads

$$\int d\Pi_2 = \int \frac{p_1^2\,dp_1}{(2\pi)^3\,4\,E_1\,E_2}\,(2\pi\,d\cos\theta_1)\;2\pi\,\delta\left(E_A+E_B-\sqrt{p_1^2+m_1^2}-\sqrt{p_1^2+m_2^2}\right)\ , \tag{13.26}$$

where we have used that $\mathbf{p_1} = -\mathbf{p_2}$ in the delta function, wich stems from the fact that we have used the spatial delta function in the center of momentum frame. We are now in a position to perform the integral in the absolute value of the spatial momentum of the particle 1, p_1, by using the delta function. Restoring the differential solid angle to have a more general expression, we have

$$\begin{aligned}\int d\Pi_2 &= \int \frac{p_1^2}{(2\pi)^2\,4E_1\,E_2}\,\frac{d\Omega_1}{|\frac{p_1}{E_1}+\frac{p_1}{E_2}|}\\ &= \int \frac{1}{16\pi^2}\,\frac{p_1}{E_1+E_2}\,d\Omega_1\ .\end{aligned} \tag{13.27}$$

But, since $E_1+E_2 = E_{\rm CM}$ then we obtain

$$\boxed{\int d\Pi_2 = \int \frac{1}{16\pi^2}\,\frac{p_1}{E_{\rm CM}}\,d\Omega_1}\ . \tag{13.28}$$

Let us compute now the cross section in the CM frame. It is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_A\,2E_B}\,\frac{1}{|v_A^z - v_B^z|}\,\frac{p_1}{16\pi^2\,E_{\mathrm{CM}}}\,|\mathcal{A}|^2\ , \tag{13.29}$$

where the solid angle refers to the final states particles, and z is the direction of the incoming A particle.
If we now consider the relative velocity we have

$$|v_A^z - v_B^z| = \left|\frac{p_A^z}{E_A} - \frac{p_B^z}{E_B}\right|\ . \tag{13.30}$$

If we now consider the simplified case $m_A = m_B = m_1 = m_2 = m$, we have

$$|v_A^z - v_B^z| = \frac{2}{E_{\mathrm{CM}}}\,|p_A^z - (-p_A^z)| = \frac{4p_A}{E_{\mathrm{CM}}} = \frac{4p_1}{E_{\mathrm{CM}}}\ , \tag{13.31}$$

Then, we arrive at a final expression for the angular distribution for scattering in the CM of two particles into two particles, all of the same mass m:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2}\,\frac{1}{E_{\mathrm{CM}}^2}\,|\mathcal{A}|^2\ . \tag{13.32}$$

13.4 Decay Rate of an Unstable Particle

If instead of considering the transition probability per unit time from a two-particle initial state we start with a state of one particle, we are computing the decay rate for the process $A \to 1\ldots n$, for the decay of a particle A to n particles in the final state. The derivation is just straightforward and the result is the differential decay probability per unit time given by

$$d\Gamma = \frac{1}{2m_A}\prod_{f=1}^{n}\left(\frac{d^3p_f}{(2\pi)^3}\,\frac{1}{2E_f}\right)(2\pi)^4\,\delta^{(4)}\left(P_A - \sum_f P_f\right)|\mathcal{A}|^2\ , \tag{13.33}$$

where the factor of $2m_A$ comes from using $2E_A$ in the rest frame of the decaying particle, and $\mathcal{A}$ is the amplitude for the decay process. For a given decay channel (i.e. a given final state), the integral gives the so-called partial width of A into that channel

$$\Gamma(A \to f_1) = \int d\Gamma(A \to f_1) \ . \tag{13.34}$$

The total width of A is a property of the particle and corresponds to the sum of the partial widths into all the available channels into which A can possibly decay

$$\Gamma_A \equiv \sum_i \Gamma(A \to f_i) \ . \tag{13.35}$$

The lifetime of tha particle is then the inverse of the total decay rate or total width. Decay rates have units of energy, thus if we want the lifetime in seconds we can use

$$\tau_A = \frac{\hbar}{\Gamma_A} \ . \tag{13.36}$$

For instance, if we initially have a given number of particles of type A, at a later time t we have

$$N(t) = N(0)\, e^{-t/\tau_A} \ . \tag{13.37}$$

The lifetime also determines the typical displacement of a particle produced before it decays. This is

$$c\,\tau_A\,\gamma \ , \tag{13.38}$$

where c is the speed of light, and γ is the relativistic factor.

Finally, the propagation of an unstable particle is affected by its decays. We will show later in the course that the propagator of a particle with open decay channels gets modified to be

$$\frac{i}{p^2 - m_A^2 - i\Gamma_A m_A} \ , \tag{13.39}$$

where we considered a scalar propagator and p is the four-momentum of A. We will derive (13.39) in the context of renormalization and see that the new term appears as a consequence of an imaginary shift in the pole of the propagator that arises due to the existence of open decay channels for A. As a result, unstable particles appear in cross sections for processes that are mediated by them as resonances of widths characterized by Γ_A. This is the reason why these particles are called resonances, and also why the total decay rate Γ_A is called the particle width.

Additional suggested readings

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 5.1.
- *Quantum Field Theory* , by M. Srednicki, Chapter 5.
- *The Quantum Theory of Fields, Vol. I*, by S. Weinberg, Section 10.3.