Lecture 12

From Correlation Functions to Amplitudes

Up to this point we have been able to compute correlation functions up to a given order in perturbation theory. Our next step is to relate these to amplitudes for physical processes that we can use to compute observables such as scattering cross sections and decay rates. For instance, assuming given initial and final states, we want to compute the amplitude to go from one to the other in the presence of interactions. In order to be able to define initial and final states we consider a situation where the interaction takes place at a localized region of spacetime and the initial state is defined at $t \to -\infty$, whereas the final state is defined at $t \to +\infty$. These asymptotic states are well defined particle states. That is, if we have a two-particle state of momentum $|p_1 p_2\rangle$, this means that the wave-packets associated with each of the particles are strongly peaked at p_1 and p_2, and are well separated in momentum space. This assumption simplifies our treatment of asymptotic states. The same will be assumed of the final state $|p_3 \dots p_n\rangle$. We will initially assume these states far into the past and into the future are eigenstates of the free theory. The fact is they are not, even if they are not interacting say through scattering. The reason is that the presence of interactions modifies the parameters of the theory, so even the unperturbed propagation from the far past or to the far future is not described by free eigenstates. However, the results we will derive for free states will still hold with some modifications associated with the shift of the paramenters of the theory (including the fields) in the presence of interactions. We will say more about this at the end of the section.

We start by defining the object the we want to compute in order to make observable predictions. The aim is to relate this observable to the correlation functions of our quantum field theory. The amplitude for a process from an initial state $|i\rangle$ of two particles of well defined momentum to go to a final state of several particles of well defined momenta is a typical observable we will need to compute to calculate cross sections and similar observables. For simplicity, we consider here a real scalar field.

For the states in the far past we define

$$|p\rangle = \sqrt{2\omega_p}\, a_p^\dagger(-\infty)|0\rangle \ , \tag{12.1}$$

where the $a_p^\dagger(-\infty)$ creates a particle of momentum $\mathbf{p}$ at $t \to -\infty$. Analogously,

$$|p\rangle = \sqrt{2\omega_p}\, a_p^\dagger(+\infty)|0\rangle \ , \tag{12.2}$$

creates a particle in the far future for $t \to +\infty$. The relativistic normalization is chosen so that the free field operator

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{+ik\cdot x} \right) \ , \tag{12.3}$$

results in

$$\langle 0|\phi(x)|p\rangle = e^{-ip\cdot x} = e^{-i\omega_p t} e^{i\mathbf{p}\cdot\mathbf{x}} \ , \tag{12.4}$$

which can be interpreted as the time-evolved, single-particle state $|p\rangle$ in position space. The amplitude then can be written as

$$\begin{aligned} \langle p_3 \dots p_n | p_1 p_2 \rangle \ &= \ \sqrt{2\omega_{p_1}} \sqrt{2\omega_{p_2}} \sqrt{2\omega_{p_3}} \dots \sqrt{2\omega_{p_n}} \\ &\times \ \langle 0| a_{p_3}(+\infty) \dots a_{p_n}(+\infty)\, a_{p_1}^\dagger(-\infty)\, a_{p_2}^\dagger(-\infty)|0\rangle \ . \end{aligned} \tag{12.5}$$

In order to relate the amplitude in (12.5) to the corresponding n-point correlation function we will need to write the creation and annihilation operators above in terms of the fields themselves.

12.1 The LSZ Reduction Formula

We want to express (12.5) in terms of fields. We start by noticing that

$$(i\partial_t + \omega_k) \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right) = 2\omega_k\, a_k\, e^{-ik\cdot x} \ . \tag{12.6}$$

When we apply this operator to the field we now have

$$\begin{aligned}\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(i\partial_t+\omega_p)\phi(x) &= \int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(i\partial_t+\omega_p)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left(a_k e^{-ik\cdot x}+a_k^\dagger e^{ik\cdot x}\right)\\ &= \int\frac{d^3k}{(2\pi)^3}\int d^3x\left(e^{-i(\mathbf{p}-\mathbf{k})\cdot x}e^{-i\omega_k t}a_k\frac{\omega_p+\omega_k}{\sqrt{2\omega_k}}+e^{-i(\mathbf{p}+\mathbf{k})\cdot x}e^{i\omega_k t}a_k^\dagger\frac{\omega_p-\omega_k}{\sqrt{2\omega_k}}\right)\,.\end{aligned}$$

Clearly, the space integrals result in factors of $(2\pi)^3\delta^{(3)}(\mathbf{p}\pm\mathbf{k})$ in both terms. Then this gives us the identity

$$\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(i\partial_t+\omega_p)\phi(x)=\sqrt{2\omega_p}\,a_p\,e^{-i\omega_p t}\,. \tag{12.7}$$

In order to derive the desired relation though, we need to compute the following

$$\begin{aligned}\int d^4x\, e^{ip\cdot x}\left(\partial^2+m^2\right)\phi(x) &= \int d^4x\, e^{ip\cdot x}\left(\partial_t^2-\nabla^2+m^2\right)\phi(x)\\ &= \int d^4x\, e^{ip\cdot x}\left(\partial_t^2+\mathbf{p}^2+m^2\right)\phi(x)\\ &= \int d^4x\, e^{ip\cdot x}\left(\partial_t^2+\omega_p^2\right)\phi(x)\,.\end{aligned} \tag{12.8}$$

To obtain the second line of (12.8) we integrated by parts (twice) the spatial derivatives so that they would act on the exponential, bringing down a $-\mathbf{p}^2$. In doing so we are still assuming, as we have done so far in the course, that sources die down for $\mathbf{x}\to\pm\infty$. On the other hand, we will not be assuming the same for $t\to\pm\infty$, since we are considering the presence of asymptotic states in the far past and far future.

Finally, we notice that the integrand in the last line of (12.8), can be rewritten as

$$e^{ip\cdot x}\left(\partial_t^2+\omega_p^2\right)\phi(x)=-i\partial_t\left[e^{ip\cdot x}\left(i\partial_t+\omega_p\right)\phi(x)\right]\,. \tag{12.9}$$

Now we can put it all together. We have from (12.8) and (12.9) that

$$\begin{aligned}\int d^4x\, e^{ip\cdot x}\left(\partial^2+m^2\right)\phi(x) &= -i\int d^4x\,\partial_t\left[e^{ip\cdot x}\left(i\partial_t+\omega_p\right)\phi(x)\right]\\ &= -i\int dt\,\partial_t\left[e^{i\omega_p t}\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\left(i\partial_t+\omega_p\right)\phi(x)\right]\\ &= -i\int dt\partial_t\left[\sqrt{2\omega_p}\,a_p\right]\,,\end{aligned} \tag{12.10}$$

where in the last equality we used (12.7). The integral is a total derivative, but it is not trivial due to the presence of the asymptotic states defined for $t \to \pm\infty$. We then obtain

$$\boxed{i\int d^4x\, e^{ip\cdot x}\left(\partial^2+m^2\right)\phi(x) = \sqrt{2\omega_p}\left[a_p(+\infty)-a_p(-\infty)\right]}\ . \tag{12.11}$$

Then we can obtain the adjoint relation from (12.11) for the creation operators

$$\boxed{-i\int d^4x\, e^{-ip\cdot x}\left(\partial^2+m^2\right)\phi(x) = \sqrt{2\omega_p}\left[a_p^\dagger(+\infty)-a_p^\dagger(-\infty)\right]}\ . \tag{12.12}$$

Now, we can use the results in (12.11) and (12.12) in the expression (12.5) for the amplitude. But first we rewrite it by using the fact that it is already time-ordered.

$$\begin{aligned}
\langle f|i\rangle &= \sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\sqrt{2\omega_{p_3}}\ldots\sqrt{2\omega_{p_n}}\\
&\times\ \langle 0|a_{p_3}(+\infty)\ldots a_{p_n}(+\infty)\,a_{p_1}^\dagger(-\infty)\,a_{p_2}^\dagger(-\infty)|0\rangle\\
&= \sqrt{2\omega_{p_1}}\ldots\sqrt{2\omega_{p_n}}\quad \langle 0|T\left(\left[a_{p_3}(+\infty)-a_{p_3}(-\infty)\right]\ldots\right. \qquad (12.13)\\
&\left.\left[a_{p_n}(+\infty)-a_{p_n}(-\infty)\right]\left[a_{p_1}^\dagger(+\infty)-a_{p_1}^\dagger(-\infty)\right]\left[a_{p_2}^\dagger(+\infty)-a_{p_2}^\dagger(-\infty)\right]\right)|0\rangle\ ,
\end{aligned}$$

where in the last equality we used the fact that the time-ordering operator T tells us to put all earlier time operators (here, those evaluated at $t \to -\infty$) to the right, whereas the later time operators should be going on the left. Since

$$a_p(-\infty)|0\rangle = 0\ , \qquad \langle 0|a_p^\dagger(+\infty) = 0\ , \tag{12.14}$$

then the equality between the first and second line in (12.13) holds. Which means that now we can rewrite the amplitude as

$$\begin{aligned}
\langle f|i\rangle &= i\int d^4x_3\, e^{ip_3\cdot x_3}\left(\partial_{x_3}^2+m^2\right)\cdots\int d^4x_n\, e^{ip_n\cdot x_n}\left(\partial_{x_n}^2+m^2\right)\\
&\times\ i\int d^4x_1\, e^{-ip_1\cdot x_1}\left(\partial_{x_1}^2+m^2\right) i\int d^4x_2\, e^{-ip_2\cdot x_2}\left(\partial_{x_2}^2+m^2\right)\\
&\times\ \langle 0|T\left(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\ldots\phi(x_n)\right)|0\rangle\ . \qquad (12.15)
\end{aligned}$$

The equation above goes by the name of LSZ formula (after Lehmann, Symanzik and Zimmermann). It gives us a relation between an amplitude involving n-particle states in momentum space and the n-point correlation function.

Although we have been considering the asymptotic fields as free fields for $t \to \pm\infty$, the LSZ reduction formula is still valid for interacting fields. The point is that we can always follow the individual particle states to the far past or the far future by making use of wave-packets and assuming that these are sufficiently separated (e.g. in momentum space). However, even if this is achieved we still need to guarantee that the following equalities are valid in the presence of interactions

$$\langle 0|\phi(x)|0\rangle = 0, \qquad \langle 0|\phi(x)|p\rangle = e^{-ip\cdot x} \ . \tag{12.16}$$

This must be assumed despite the fact that in general $\phi(x)$ is not given by (12.3) anymore. The way to ensure this is to redefine the field in the presence of interactions in order to satisfy (12.16). This field redefinition in the presence of interactions is part of a procedure called renormalization, and it also affects the mass and the interaction coupling. We will go back to this point later in the course. But the point is that even after these redefinitions, the LSZ reduction formula is valid since the analytic structure of the (renormalized) correlation functions regarding the asymptotic states (e.g. the pole structure of the correlation function) remains the same (accounting for the shift in the mass, couplings and field).

The meaning of the LSZ reduction formula becomes clearer once we apply the Klein-Gordon operators on the correlation functions. Each of these operators acts on a field evaluated on an external point x_i, with $i = 1, 2, \ldots$. But for each of these we have

$$(\partial^2_{x_i} + m^2)\, D_F(x_i - y) = -i\delta^{(4)}(x_i - y) \ , \tag{12.17}$$

where y is some other point (external or internal). Thus, we see that each of the operators turns the external propagators into delta functions. This is sometimes called "amputation": in order to obtain the actual amplitude the external propagators are removed in the way prescribed by (12.15). This is the result of the fact that the external states are *on shell*. As we will see below, computing an amplitude requires to obtain the connected diagrams from the appropriate *connected* correlation function to the desired order on perturbation theory, and then *amputate* it of the external propagators. We will illustrate this with some examples.

Example 1: We want to compute the four-point function up to order λ in the ϕ^4 theory of a real scalar field. In particular, we want to compute an amplitude for scattering two particles of momenta p_1 and p_2 into two final state particles of momenta p_3 and p_4

$$\langle p_3 p_4 | p_1 p_2\rangle \ . \tag{12.18}$$

As we have seen before, the order λ^0 contributions are disconnected diagrams only. They are

$$\begin{aligned} G_0^{(4)}(x_1,x_2,x_3,x_4) &= D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4) \\ &\quad +D_F(x_1-x_4)D_F(x_2-x_3) \qquad (12.19) \end{aligned}$$

When we apply (12.15) to get the amplitude (12.18) we get no contribution to this order. The reason is that the number of poles in the above expression is not enough to kill the "zeroes" in the LSZ formula coming from the four operators $(\partial^2_{x_i}+m^2)$. These zeroes are associated to the action of the two remnant Klein-Gordon operators acting on the delta functions that resulted from using (12.17) on the other two propagators in (12.19), and they basically correspond to the on-shellness conditions $-p_i^2+m^2=0$ for all the asymptotic states. One can check that the same will happen to any disconnected diagram. As a conclusion, we see that disconnected diagrams in the four-point function do not contribute to the scattering amplitude (12.18).

On the other hand, going to order λ we have the *fully connected* four-point function (see previous lectures)

$$G_\lambda^{(4)}(x_1,x_2,x_3,x_4) = (-i\lambda)\int d^4y\, D_F(x_1-y)\, D_F(x_2-y)\, D_F(x_3-y)\, D_F(x_4-y)\ . \quad (12.20)$$

The application of the LSZ reduction formula (12.15) to the expression above results in

$$\begin{aligned} \langle p_3p_4|p_1p_2\rangle &= i\int d^4x_1\, e^{-ip_1\cdot x_1}\left(\partial^2_{x_1}+m^2\right) i\int d^4x_2\, e^{-ip_2\cdot x_2}\left(\partial^2_{x_2}+m^2\right) \qquad (12.21) \\ &\quad i\int d^4x_3\, e^{ip_3\cdot x_3}\left(\partial^2_{x_3}+m^2\right) i\int d^4x_4\, e^{ip_4\cdot x_4}\left(\partial^2_{x_4}+m^2\right)\, G_\lambda^{(4)}(x_1,x_2,x_3,x_4)\ . \end{aligned}$$

Applying (12.21) to (12.20) we obtain

$$\begin{aligned} \langle p_3p_4|p_1p_2\rangle &= (-i\lambda)\int d^4y \int d^4x_1\, e^{-ip_1\cdot x_1}\delta^{(4)}(x_1-y)\int d^4x_2\, e^{-ip_2\cdot x_2}\delta^{(4)}(x_2-y) \\ &\times \int d^4x_3\, e^{ip_3\cdot x_3}\delta^{(4)}(x_3-y)\int d^4x_4\, e^{ip_4\cdot x_4}\delta^{(4)}(x_4-y) \\ &= (-i\lambda)\int d^4y\, e^{-i(p_1+p_2-p_3-p_4)\cdot y} \\ &= (-i\lambda)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4) \qquad (12.22) \end{aligned}$$

From this expression, we see that the amplitude is just the insertion of the vertex factor $(-i\lambda)$ times a momentum conservation delta function. The appearance of this delta function is associated to the fact that all external points are connected to the same internal point y where the interaction takes place. That is, it comes from the fact that the interaction is local. Another important point is that, unlike for the order λ^0 above, the singularities of the contribution to the four-point function $G^{(4)}_\lambda(x_1, x_2, x_3, x_4)$ exactly match the action of the Klein-Gordon operators in (12.21). The result above is a first example of a Feynman rule in momentum space. Insert the interaction factor $(-i\lambda)$ and a momentum conservation delta function in each vertex. Strip all external propagators (which is the result of applying the LSZ formula). This is schematically shown in Figure 12.1 below.

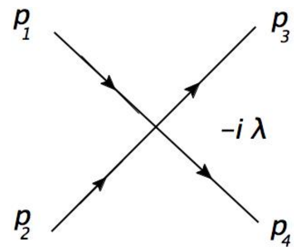

Figure 12.1: Momentum-space Feynman rule for the four-point amplitude to order λ in ϕ^4 theory.

However, there is more to the Feynman rules. To derive the rest we need to go to an example with higher orders in λ.

Example 2: Four-point function to order λ^2.
The λ^2 contribution to the four-point function can be obtained from

$$G^{(4)}_{\lambda^2}(x_1, x_2, x_3, x_4) = \frac{1}{Z[0]} \int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \times \frac{1}{2!} \frac{(-i\lambda)^2}{(4!)^2} \int d^4y\, \phi^4(y) \int d^4z\, \phi^4(z)\ . \tag{12.23}$$

In (12.23), the factor of 1/2! coming from the exponential expansion cancelled by the exchange $y \leftrightarrow z$. We will concentrate on connected diagrams. There are three ways of connecting the external fields to the eight fields at points y and z of the interactions. They are depicted in Figure 12.2 below.

Let us focus on the first diagram (a). The combinatoric factor in front of it can be obtained by counting the ways to match $\phi(x_1)$ with $\phi(y)$ (4), times the ways of matching $\phi(x_2)$ with the remaining $\phi(y)$ (3), times the 4 ways of matching $\phi(x_3)$ with $\phi(z)$, times the 3 ways to match $\phi(x_4)$ with $\phi(z)$. Finally, we need to contract the remaining $\phi(y)$ and $\phi(z)$, which brings an extra factor of 2. All in all, the combinatoric factor times $1/(4!)^2$ results in an overall factor of

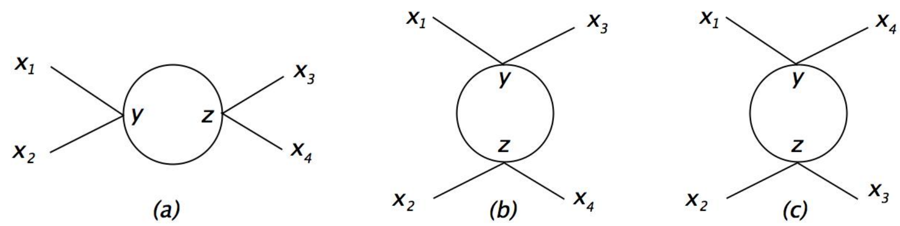

Figure 12.2: Connected diagrams contributing to the four-point function to order λ^2 in ϕ^4 theory.

$$(-i\lambda)^2 \frac{1}{2} \ . \tag{12.24}$$

We can understand the factor of 1/2 above in this diagram as a *symmetry factor*. It is the factor we need to divide by if we assume that at each vertex of the diagram we insert a factor of $-i\lambda$, which is the coefficient for the four-point function at order λ. In this diagram, using $-i\lambda$ at each vertex is overcounting the combinatoric factor since it is tantamount to assuming that all the lines at the two vertices are *un-contracted* fields. But we know that the internal lines coming from the vertices result in contractions into two propagators. To obtain the symmetry factors we see that the use of $-i\lambda$ will result in counting diagrams interchanging the internal integration points y and z as distinct contributions. But this is not the case. So we can think of this factor of 2 as obtained by exchanging the two internal propagators, resulting in undistinguishable contributions.

The result for the contribution to the four-point function is

$$\begin{aligned} G^{(4)}_{(a)}(x_1,x_2,x_3,x_4) &= \frac{(-i\lambda)^2}{2} \int d^4y\, d^4z\, D_F(x_1-y)\, D_F(x_2-y)\, D_F(x_3-z)\, D_F(x_4-z) \\ &\qquad D_F(y-z) D_F(y-z) \ . \end{aligned} \tag{12.25}$$

We want to obtain the $\mathcal{O}(\lambda^2)$ contributions to the scattering amplitude for two particles of initial fixed momenta to go to other two particles of known final momenta. Applying (12.15) on (12.25) we get

$$\langle p_3 p_4 | p_1 p_2 \rangle_{(a)} = \frac{(-i\lambda)^2}{2} \int d^4y\, d^4z\, e^{-i(p_1+p_2)\cdot y}\, e^{i(p_3+p_4)\cdot z}\, D_F(y-z)\, D_F(y-z) \ , \tag{12.26}$$

where the action of each Klein-Gordon operator $(\partial^2_{x_i} + m^2)$ on the propagators containing an external point x_i in the argument resulted in factors of $-i\delta^{(4)}(x_i - y)$ and $-i\delta^{(4)}(x_i - z)$ which we used to integrate over the x_i's. Since the two internal propagators do not have external positions in their arguments they remain in (12.26). In order to make further progress we are going to express these propagators in momentum space by making use of

$$D_F(y-z) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}\, \frac{i}{q^2-m^2+i\epsilon}\ , \tag{12.27}$$

in (12.26). We then obtain

$$\begin{aligned}\langle p_3p_4|p_1p_2\rangle_{(a)} &= \frac{(-i\lambda)^2}{2}\int d^4y\, d^4z\, e^{-i(p_1+p_2)\cdot y}\, e^{i(p_3+p_4)\cdot z} \\ &\times \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}\, \frac{i}{q^2-m^2+i\epsilon} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(y-z)}\, \frac{i}{k^2-m^2+i\epsilon}\ .\end{aligned}$$

We can now perform the integrals in the positions y and z resulting in two delta functions. This gives

$$\begin{aligned}\langle p_3p_4|p_1p_2\rangle_{(a)} &= \frac{(-i\lambda)^2}{2}\int \frac{d^4q}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\, (2\pi)^4\, \delta^{(4)}(p_1+p_2+q+k)\, (2\pi)^4\, \delta^{(4)}(p_3+p_4+q+k) \\ &\times \frac{i}{q^2-m^2+i\epsilon}\, \frac{i}{k^2-m^2+i\epsilon}\end{aligned} \tag{12.28}$$

Performing one of the momentum integrals above by using one of the delta functions (e.g. integrating over k) we finally obtain

$$\begin{aligned}\langle p_3p_4|p_1p_2\rangle_{(a)} &= \frac{(-i\lambda)^2}{2}\, (2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4) \\ &\times \int \frac{d^4q}{(2\pi)^4}\, \frac{i}{q^2-m^2+i\epsilon}\, \frac{i}{k^2-m^2+i\epsilon}\ ,\end{aligned} \tag{12.29}$$

where the value of k is fixed at

$$k = -p_1 - p_2 - q = -p_3 - p_4 + q\ , \tag{12.30}$$

by the integration with the delta functions in (12.28). As we can see from (12.29), there remains an undetermined momentum q which must be integrated over. This can be easily understood by looking at the diagram once again, now in momentum space and with all these momenta drawn explicitly. It is clear that, although we have two internal lines, only one of the two internal momenta are independent: there is a delta function forcing

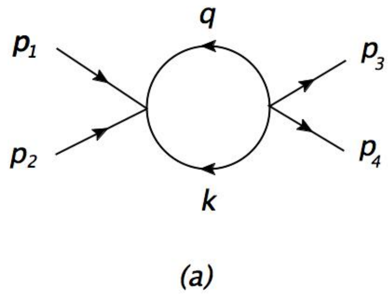

Figure 12.3: One of the Feynman diagrams in momentum space for the four-point amplitude to order λ^2 in ϕ^4 theory.

momentum conservation at each vertex, but the overall momentum conservation is not a constraint so there is one undetermined momentum we still have to integrate over. Diagrams (b) and (c) in Figure 12.2 are very similar and are left as an exercise.

Finally, we should consider the $O(\lambda^2)$ connected diagramas such as the one in Figure 12.4. Here we see a loop correction to $O(\lambda)$ to one of the external lines.

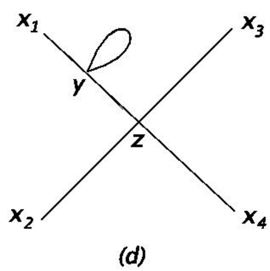

Figure 12.4: One of the Feynman diagrams in position space for the four-point amplitude to order λ^2 in ϕ^4 theory. It corresponds to an $O(\lambda)$ correction to one of the external lines.

There are obviously three other similar diagrams. Let us compute its contribution to the amplitude $\langle p_3 p_4 | p_1 p_2 \rangle$. For this, we make use of the LSZ formula applied to the contribution to the four-point correlation function to this order

$$\begin{aligned} G^{(4)}(x_1,\ldots,x_4)_{(d)} &= \frac{(-i\lambda)^2}{2}\int d^4y\; d^4z\, D_F(x_1-y)\, D_F(y-z)\, D_F(y-y)\, D_F(x_2-z) \\ &\quad \times D_F(x_3-z)\, D_F(x_4-z)\ . \qquad (12.31) \end{aligned}$$

Following the same procedure that lead us to (12.29), we obtain

$$\langle p_3p_4|p_1p_2\rangle_{(d)} = \frac{(-i\lambda)^2}{2}(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4) \times \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\,\frac{i}{k^2-m^2+i\epsilon}\,, \tag{12.32}$$

but where now we have

$$k = p_2 - p_3 - p_4 = -p_1 \ . \tag{12.33}$$

The corresponding Feynman diagram in momentum space is shown in Figure 12.5.

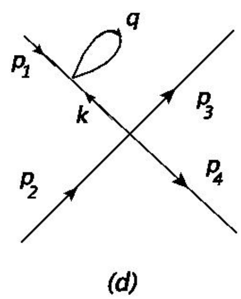

Figure 12.5: One of the Feynman diagrams in momentum space for the four-point amplitude to order λ^2 in ϕ^4 theory. It corresponds to an $O(\lambda)$ correction to one of the external lines, which is on shell.

But now we see that since the external legs are all on shell, we have

$$k^2 = p_1^2 = m^2 \ , \tag{12.34}$$

which means that the contribution $\langle p_3p_4|p_1p_2\rangle_{(d)}$ is *divergent*, as can be seen from (12.32). Clearly, this is also the case with the other diagrams of this type corresponding to $O(\lambda)$ corrections of the other three external legs. The reason is the existence of an internal propagator whose momentum goes on shell (e.g. $k^2 = m^2$ above). The same will be true of the corrections to external legs of any diagram and up to any order in perturbation theory. These are in fact part of the *renormalization* of the external legs of any diagram and should not be considered when computing an amplitude. These diagrams should be excluded from the calculation, since they are going to be included by the renormalization process, which redefines the fields (leading in this case to the redefinition of the propagators) in the presence of interactions. We could imagine including these diagrams as in Figure 12.6,

We can interpret the effects of these diagrams in a completely analogous way to the shift of the vacuum in the presence of the interactions, $|0\rangle \to |\tilde{0}\rangle$. Here we are considering an

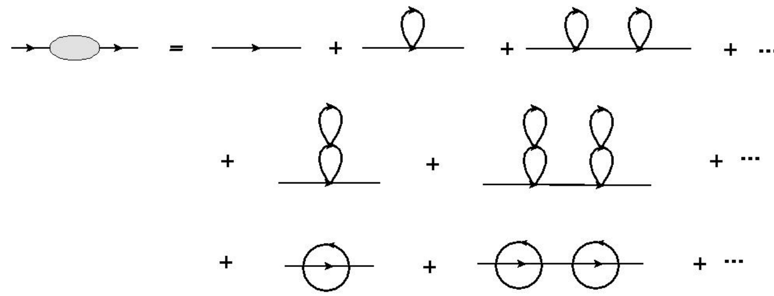

Figure 12.6: Corrections to the external lines to all orders in perturbation theory. The result in a redefinition of the propagator, which is absorbed as a redefinition of the fields entering the two-point function.

asymptotic state which is not a free state of momentum p, i.e. there is a correction that we can schematically write as

$$|p\rangle \to |\tilde{p}\rangle = \sqrt{Z}\,|p\rangle \ , \tag{12.35}$$

where the factor $\sqrt{Z}$ comes from the corrections to the propagation of the asymptotic state of momentum p. But these corrections do not affect the calculation of the scattering amplitudes we are after. When computing an amplitude to a certain order in perturbation theory, we need to include the asymptotic states corrected to the appropriate level. So these diagrams should not be explicitly present in the computation. The procedure to exclude these diagrams, which can be disconnected from the rest by cutting just one propagator line, is called *amputation*. To amputate a diagram is to go from its outer lines inward up to the last propagator that is always on shell by virtue of momentum conservation, and to cut this propagator line. In the diagram of Figure 12.6 the procedure simply corresponds to cut the diagram at the propagator with momentum k. This leaves us with an order λ contribution to the four-point function contributing to the amplitude, plus a propagator with an $O(\lambda)$ correction, as seen in Figure 12.7. The latter, the amputated correction, does not enter in the calculation of the amplitude. Thus, the Feynman diagrams in momentum space that actually contribute to the computation of the scattering amplitude, are the *connected* and *amputated* diagrams.

12.2 Feynman Rules in Momentum Space

From the examples in the previous section we can conclude that we can write down the contribution to the amplitude of a process to a given order in perturbation theory by following a simple set of rules. First, we need to select the *connected* and *amputated* Feynman diagrams up to the desired order in perturbation theory. Then, we compute

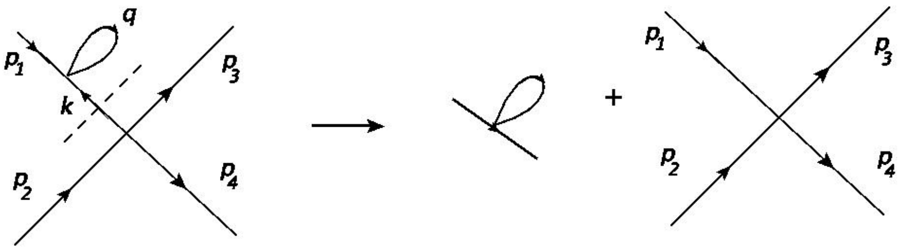

Figure 12.7: Amputation of a diagram to obtain a connected and amputated diagram contributing to the scattering amplitude. The dashed line on the diagram on the left corresponds to the "amputating" cut.

the contribution of these diagrams following the Feynman rules, which for the real scalar theory with a ϕ^4 interaction we have been studying are:

1. Vertex: Insert a factor of $-i\lambda$ for each vertex in the diagram. It is clear from (12.29) that this will get us the factor in front up to symmetry factors. Notice that this is the Feynman rule of the diagram at order λ (i.e. at "tree" level and without "loops"). In general, deriving the tree-level interaction vertex is one of the first things we need to do in a theory in order to be able to obtain its Feynman rules.

2. Momentum conservation at each vertex: The presence of the delta functions at each vertex in (12.28) tells us that momentum conservation must be enforced at each vertex in the diagram. This always results in an overall delta function enforcing total momentum conservation. For our case is the factor $(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$.

3. Loop momentum integration: Integrate over all the undetermined momenta. In our example from Figure 12.3 , the product of the two delta functions in (12.28) is equivalent to the overall momentum conservation. So one of the two internal momenta remains free and must be integrated over.

4. Symmetry factors: We must divide by the symmetry factor of the diagram. In Figure 12.3 this is 2, since the internal propagators can be exchanged without consequence. The need to divide by the symmetry factors stems from the fact that in any generic diagram we use the vertex Feynman rule (here $-i\lambda$) for each interaction. But generally this has the correct combinatoric factor only in the tree-level interaction vertex. This "mistake" must be corrected by the symmetry factor.

Additional suggested readings

- *Quantum Field Theory*, by C. Itzykson and J. Zuber, Chapter 5.1.
- *Quantum Field Theory* , by M. Srednicki, Chapter 5.
- *The Quantum Theory of Fields, Vol. I*, by S. Weinberg, Section 10.3.