Lecture 11

Functional Integral for Fermions

Up until this point, we derived methods that allow us to compute correlation functions of fields obeying commutation relations. To derive similar methods to be applied to fermions we need a bit more work. In particular we need to define Grasmann variables and their integration rules before we can write down a meaningful functional integral for fermions.

11.1 Grassmann Variables

Grassmann numbers are defined by their anti-commutation. That is given two Grassmann numbers θ and η, we have

$$\theta\eta = -\eta\theta \ . \tag{11.1}$$

In particular, this means that given an anti-commuting number θ, it satisfies

$$\theta^2 = 0 \ . \tag{11.2}$$

Based on this simple properties it is possible to derive many others. For instance,

$$(\theta\eta)\,\phi = -\theta\phi\eta = +\phi(\theta\eta) \ , \tag{11.3}$$

meaning that the product of two Grassmann numbers *commutes* with other Grassmann numbers. In particular, is it of interest to point out that the most general function of a Grassmann variable θ can be written as

$$f(\theta) = A + B\theta \ , \tag{11.4}$$

where A and B are normal constants, that is it is at most linear in θ due to (11.2).

Integration of Grassmann Variables:

We would like to define the integral of a function of a Grassmann variable. This is

$$\int d\theta\, f(\theta) = \int d\theta\, (A + B\theta) \ . \tag{11.5}$$

We want the integration to be invariant under a constant shift. That means that it remains the same under the change of variables

$$\theta \longrightarrow \theta + \eta \ , \tag{11.6}$$

where η is a constant Grassmann number. This gives

$$\int d\theta (A + B\theta) \longrightarrow \int d\theta (A + B\eta + B\theta) \ . \tag{11.7}$$

For these two expressions to be the same we need the second term on the right to vanish, which requires

$$\int d\theta = 0 \ . \tag{11.8}$$

Finally, we need to define

$$\int d\theta\, \theta = C \ , \tag{11.9}$$

where C is a constant number. We choose $C = 1$ for simplicity. So in sum, we have

$$\int d\theta = 0 \ , \qquad\qquad \int d\theta\, \theta = 1 \ . \tag{11.10}$$

For complex Grassmann variables, θ and θ^*, defined by

$$\theta \equiv \frac{\theta_1 + i\theta_2}{\sqrt{2}} \ , \qquad\qquad \theta^* \equiv \frac{\theta_1 - i\theta_2}{\sqrt{2}} \ , \tag{11.11}$$

we conveniently define the complex conjugation of a product as

$$(\theta\,\eta)^* \equiv \eta^*\,\theta^* = -\theta^*\,\eta^* \ , \tag{11.12}$$

which then results in the last equality above, as we should expect. So we see that the complex conjugation here is like the hermitian conjugation of operators.

Of particular interest to us, is the fact that

$$d\theta_1\,d\theta_2 = i\,d\theta\,d\theta^* \ . \tag{11.13}$$

Thus, we can use θ and θ^* as independent variables.

11.2 Gaussian Integrals in Complex Grassmann Variables

We start with the integral

$$\begin{aligned} \int d\theta^*\,d\theta\,e^{-\theta^*\,b\,\theta} &= \int d\theta^*\,d\theta(1-\theta^*\,b\,\theta) \ , \\ &= \int d\theta^*\,d\theta\,(1+\theta\,\theta^*\,b) = b \ , \end{aligned} \tag{11.14}$$

where we used the convention that the relative sign between the two integrals is $+1$. So we arrive at

$$\boxed{\int d\theta^*\,d\theta\,e^{-\theta^*\,b\,\theta} = b} \ . \tag{11.15}$$

It is interesting to contrast the result above with the one we would obtain if θ was a normal complex number ($2\pi/b$). Another integral of interest will be

$$\begin{aligned} \int d\theta^*\,d\theta\,\theta\,\theta^*\,e^{-\theta^*\,b\,\theta} &= \int d\theta^*\,d\theta\,\theta\,\theta^*\,(1-\theta^*\,b\,\theta) \\ &= \int d\theta^*\,d\theta\,\theta\,\theta^*\,(1+\theta^*\,\theta\,b) = 1 \ . \end{aligned} \tag{11.16}$$

If we now consider a multidimensional integral of the type

$$I = \left(\prod_i \int d\theta_i^* \, d\theta_i\right) e^{-\theta_i^* \, B_{ij} \, \theta_j} \ , \tag{11.17}$$

where B is a hermitian matrix with eigenvalues b_i. We can rotate the θ_i's and the θ_i^*'s by a unitary transformation that diagonalizes B, defined by

$$\begin{aligned} \theta_i^* &\rightarrow U_{ki}^* \, \theta_k'^* \\ \theta_i &\rightarrow U_{j\ell} \, \theta_\ell' \ , \end{aligned} \tag{11.18}$$

we can then write

$$\begin{aligned} I &= \left(\prod_i \int d\theta_i^* \, d\theta_i\right) e^{-\sum_i \theta_i^* \, b_i \, \theta_i} \\ &= \left(\prod_i \int d\theta_i^* \, d\theta_i \, e^{-\theta_i^* \, b_i \, \theta_i}\right) = \prod_i b_i \\ &= \det B \ , \end{aligned} \tag{11.19}$$

where in the last equality we used the result from (11.15).

11.3 Functional Integral for Fermions

We can finally write down the functional integral for free fermion fields. First, we notice that the generating functional for fermion fields in the presence of sources requires two of them: one for ψ and another one for $\bar{\psi}$. This gives the generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, e^{i \int d^4x \left\{\bar{\psi}(x) \, \mathcal{O}_x \, \psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right\}} \ , \tag{11.20}$$

where the Dirac operator is $\mathcal{O}_x = i \, \not{\partial}_x - m$, and η and $\bar{\eta}$ are the linearly coupled sources. If in particular we want to compute the generating functional without sources is given by (using (11.19)

$$Z[0,0] = \int \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, e^{i \int d^4x \, \bar{\psi} \, \mathcal{O} \, \psi} \ = N \ \det \mathcal{O} \ , \tag{11.21}$$

unlike the bosonic case, where the sourceless generating functional goes like $1/\sqrt{\det \mathcal{O}}$. Finally, in complete analogy with the bosonic case, we can prove that we can define

$$\psi'(x) = \psi(x) + i \int d^4y \, S_F(x-y) \, \eta(y) \tag{11.22}$$

and

$$\bar{\psi}'(x) = \bar{\psi}(x) + i \int d^4y \, \bar{\eta}(y) \, S_F(y-x) \ , \tag{11.23}$$

such that when we make the replacement in the original functional integral in (11.20) and assume that

$$\mathcal{D}\bar{\psi}' \, \mathcal{D}\psi' = \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \ , \tag{11.24}$$

then we obtain

$$Z[\eta, \bar{\eta}] = Z[0,0] \, e^{-\int d^4x \, d^4y \, \bar{\eta}(x) \, S_F(x-y) \, \eta(y)} \ , \tag{11.25}$$

where $S_F(x-y)$ satisfies

$$\mathcal{O}_x \, S_F(x-y) = i\delta^{(4)}(x-y) \ . \tag{11.26}$$

Of course, $S_F(x-y)$ is the fermion propagator, and we can easily show that in momentum space it should be

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot x} \, \frac{i}{\not{k} - m + i\epsilon} \ . \tag{11.27}$$

Additional suggested readings

- *An Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Chapter 9.5.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter 2.5, first few sections.