Lecture 10

Interactions and Feynman Diagrams

We will start considering a theory with interactions with lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int.}} \tag{10.1}$$

where the first term $\mathcal{L}_0$ refers to the free lagrangian and the sencond one $\mathcal{L}_{\text{int.}}$ contains the interaction terms. For instance, for a real scalar theory we have

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \ . \tag{10.2}$$

The generating functional in the presence of a linearly coupled source $J(x)$ now is

$$Z[J] = N\int \mathcal{D}\phi \ e^{i\int d^4x\,\{\mathcal{L}_0+\mathcal{L}_{\text{int.}}+J(x)\phi(x)\}} \ , \tag{10.3}$$

Defining the generating functional for the free theory as

$$Z_0[J] \equiv N\int \mathcal{D}\phi \ e^{i\int d^4x\,\{\mathcal{L}_0+J(x)\phi(x)\}} \ , \tag{10.4}$$

we can rewrite (10.3) as

$$Z[J] = e^{i\int\, d^4x\,\mathcal{L}_{\text{int.}}[-i\frac{\delta}{\delta J(x)}]} \ Z_0[J] \ . \tag{10.5}$$

In the expression above $\mathcal{L}_{\text{int.}}[-i\delta/(\delta J(x))]$ means that the argument of the functional $\mathcal{L}_{\text{int.}}[\phi(x)]$ is obtained by functional derivative in the following way

$$-i\frac{\delta}{\delta J(x)}\,Z_0[J] = N\int \mathcal{D}\phi(x)\,\phi(x)\,e^{i\int d^4x\{\mathcal{L}_0 + J(x)\phi(x)\}} \tag{10.6}$$

which can be readily checked from (10.4). Then we can verify that expanding the exponential in (10.5)

$$\begin{aligned} Z[J] \;=\; & \Big(1 + i\int d^4x\,\mathcal{L}_{\rm int.}[-i\frac{\delta}{\delta J(x)}] \\ & + \frac{(i)^2}{2!}\int d^4x\,\mathcal{L}_{\rm int.}[-i\frac{\delta}{\delta J(x)}]\int d^4y\,\mathcal{L}_{\rm int.}[-i\frac{\delta}{\delta J(y)}] + \ldots\Big)\,Z_0[J]\;, \end{aligned} \tag{10.7}$$

we reobtain (10.3).

Perturbation theory means here that we assume that the interaction is a perturbation and we can only take the first few terms as a very good approximation. More precisely, we assume that there is a parameter in $\mathcal{L}_{\rm int.}$ whose increasing powers in the higher order terms in (10.7) suppress them relative to lower order terms. These are the basis for perturbation theory in the functional integral. As an example let us consider

$$\mathcal{L}_{\rm int.} = -\frac{\lambda}{4!}\phi^4 \tag{10.8}$$

This results in

$$Z[J] = e^{-i\frac{\lambda}{4!}\int d^4z\left(-i\frac{\delta}{\delta J(z)}\right)^4}\,Z_0[J]\;. \tag{10.9}$$

Remembering from the previous lecture that

$$Z_0[J] = Z_0[0]\,e^{-\frac{1}{2}\int d^4x\int d^4y J(x)\,D_F(x-y)\,J(y)}\quad, \tag{10.10}$$

we are now in a position to compute any correlation function by using

$$G^{(n)}(x_1,\ldots,x_n) = (-i)^n\frac{1}{Z[0]}\,\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)}Z[J]\Big|_{J=0}\quad, \tag{10.11}$$

applied to (10.9). For instance, let us compute the two-point function at order λ in perturbation theory:

$$
\begin{aligned}
G^{(2)}(x_1,x_2) &= \frac{(-i)^2}{Z[0]}\left(\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\right) Z[J]\Big|_{J=0} \qquad (10.12)\\
&= \frac{(-i)^2}{Z[0]}\left(\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\right)\left(1-i\frac{\lambda}{4!}\int d^4z\left(-i\frac{\delta}{\delta J(z)}\right)^4+\dots\right) Z_0[J]\Big|_{J=0}\ .
\end{aligned}
$$

First of all, we notice that here $Z[0] \neq Z_0[0]$, since they differ by terms starting at order λ:

$$
\begin{aligned}
Z[0] &= Z_0[0] + N\int \mathcal{D}\phi\, e^{i\int d^4z\,\mathcal{L}_0[\phi]}\, i\left(-\frac{\lambda}{4!}\right)\int d^4x\,\phi(x)^4\\
&\quad + N\int \mathcal{D}\phi\, e^{i\int d^4z\mathcal{L}_0[\phi]}\,\frac{i^2}{2!}\left(\frac{-\lambda}{4!}\right)^2\int d^4x\phi^4(x)\int d^4y\phi^4(y)+\dots\ . \qquad (10.13)
\end{aligned}
$$

Remembering that $Z_0[0] = \langle 0|0\rangle$ is the normalization of the vacuum, we can interpret $Z[0]$ as the normalization of the new vacuum in the presence of interactions, $|\tilde{0}\rangle$. Then (10.13) corresponds to

$$
\langle\tilde{0}|\tilde{0}\rangle = \langle 0|0\rangle + \dots\ , \qquad (10.14)
$$

where the dots correspond to the λ-dependent terms in (10.13).

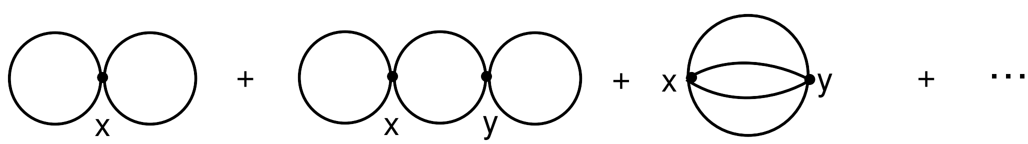

Figure 10.1: Corrections to the vacuum state coming from the interactions. The first two bubbles are the order λ, whereas the third and fourth diagrams are the λ^2 corrections appearing in $Z[0] - Z_0[0]$.

We can represent these corrections diagrammatically thanks to Wick's theorem. The first correction corresponds to the product of two propagators such as

$$
\int d^4x\, D_F(x-x)\, D_F(x-x)\ , \qquad (10.15)
$$

and is represented by the first diagram in Figure 10.1. The term of order λ^2 in the second line of (10.13) will result in the products of four propagators giving terms such as

$$
\int d^4x \int d^4y\, D_F(x-x)\, D_F(x-y)\, D_F(x-y)\, D_F(y-y)\ , \qquad (10.16)
$$

as represented in the third diagram in Figure 10.1, or in the following combination

$$\int d^4x \int d^4y\, D_F(x-y)\, D_F(x-y)\, D_F(x-y)\, D_F(x-y)\ , \tag{10.17}$$

as represented by the last diagram of Figure 10.1. As we will see later in this lecture, the role of the *denominator* $Z[0]$ as a correction of the vacuum will be to cancel the disconnected diagrams in the *numerator* of the correlation functions.

Now, going back to the two-point function in (10.12), and neglecting the denominator $Z[0]$ for the moment (i.e. assuming $Z[0] = Z_0[0]$ or $|\tilde{0}\rangle = |0\rangle$), we obtain the relevant contributions $G^{(2)}(x_1, x_2)$ by focusing on the terms of the free generating functional $Z_0[J]$ in (10.10) that contain 6 sources or three propagators, since we have six functional derivatives with respect to source at various points. There will be two possible topologies associated with the diagrams shown in Figure 10.2

Figure 10.2: Order λ corrections to the two-point function in the theory described in the text.

The disconnected diagram has a combinatoric factor of $6 \times 4!$, whereas the connected one has a factor of $6 \times 4 \times 4!$, resulting in

$$\begin{aligned} G^{(2)}(x_1, x_2) &= D_F(x_1 - x_2) - i\lambda \left(\frac{1}{8} D_F(x_1 - x_2) \int d^4z\, D_F(z-z)\, D_F(z-z) \right. \\ &\left. + \frac{1}{2} \int d^4z\, D_F(x_1 - z)\, D_F(x_2 - z)\, D_F(z-z) \right) . \end{aligned} \tag{10.18}$$

As we will formally show in the last section, the *disconnected* diagram in Figure 10.2 does not constitute a correction ot the two-point function *per se* but it is a correction to the vacuum state of the free theory resulting from the interactions at order λ. As such, it will be cancelled when we consider the normalization using the correct vacuum $|\tilde{0}\rangle$ (i..e. dividing by $Z[0]$ instead of by $Z_0[0]$).

Similarly, we can compute the four-point function. In this case, at order λ we will have a total of eight functional derivatives acting of $Z_0[J]$. Focusing on the diagrams where each of the external four points x_1, x_2, x_3, x_4 are connected to an internal point z, we have a contribution to the four-point function to order λ which is given by

$$-i\frac{\lambda}{4!}\, 8\times 6\times 4\times 2\times 4!\left(\frac{-1}{2}\right)^4 \frac{1}{4!}\int d^4z\, D_F(x_1-z)\, D_F(x_2-z)\, D_F(x_3-z)\, D_F(x_4-z)$$
$$= (-i\lambda)\int d^4z\, D_F(x_1-z)\, D_F(x_2-z)\, D_F(x_3-z)\, D_F(x_4-z)\ . \tag{10.19}$$

Notice that the combinatoric in these diagrams is such that all factors cancel. The factor $8\times 6\times 4\times 2$ corresponds to the number of ways to pick sources out of the eight that are not paired by a propagator. The first 4! factor corresponds to the remaining choices for the derivatives with respect to $J(z)$. Finally, there is an additional denominator of 4! coming from the fact that we picked the fourth term in the expansion of the exponential in (10.10). This corresponds to the connected diagram

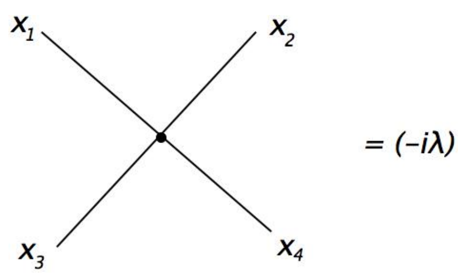

Figure 10.3: Connected diagram contribution to the four-[point function to order λ.

The connected diagram of Figure 10.3 will be useful to define the insertion of the interaction in several diagrams. For instance, if in the diagrams of (10.2) we insert $-i\lambda$ for each occurrence of the four-point vertex, we would be making a mistake of a factor of 8 in the first diagram. This can be attributed to the symmetry of the bubble type diagram in which we can perform two mirror images times an overall "up-down" switch, resulting in a factor of $2\times 2\times 2 = 8$ denominator. Similarly, in the second diagram of Figure 10.2 we have a factor of 2 coming from the mirror switch of the bubble in the diagram. These factors in denominators, of 8 and 2 respectively in the diagrams of Figure 10.2 are called symmetry factors. They are the factors we have to divide each diagram by when considering that the insertion of the vertex is given merely by $-i\lambda$.

Of course, there are many more diagrams for the four-point function at order λ. But they are all disconnected. The diagram on the left in Figure 10.4 is a vacuum correction, i.e. the free propagation is unaffected by the interaction, but the ground state gets a correction of order λ. This diagram will be cancelled by the use of $Z[0]$ in the denominator as it is a correction to the vacuum of the theory.

Figure 10.4: Disconnected order λ contributions to the four-point function.

On the other hand, the diagram on the right corresponds to a correction of order λ to one of the propagators. So it would be already taken into account if we assume the corrected propagator to $\mathcal{O}(\lambda)$. This is the content of the discussion about connected vs. disconnected Feynman diagrams that we will have in the next section.

Higher orders in λ, i.e. higher orders in the insertion of the interaction, can be easily obtained. For instance, for the order λ^2 corrections to the two-point function we need to consider 10 functional derivatives, or 5 propagators. Diagrams look like those in Figure 10.5, where the one on the left is a connected order λ^2 correction to the propagator, whereas the one on the right is a correction to the vacuum.

Figure 10.5: Order λ^2 corrections to the two-point function.

10.1 Perturbation Theory in the Correlation Function

In the previous section we implemented perturbation theory in the generating functional, such as in (10.7). However, it is also possible to implement it directly in the correlation functions. For this purpose we need to start by expanding the linear source term in (10.3) obtaining

$$\begin{aligned}
Z[J] &= N \int \mathcal{D}\phi \; e^{i\int d^4x\,\{\mathcal{L}_0+\mathcal{L}_{\rm int.}\}} \left(1 + i\int d^4x J(x)\,\phi(x)\right. \\
&+ \frac{i^2}{2!}\int d^4x_1\, d^4x_2 J(x_1)J(x_2)\phi(x_1)\phi(x_2) \\
&\left. + \dots + \frac{i^n}{n!}\int d^4x_1 \dots d^4x_n\, J(x_1)\dots J(x_n)\,\phi(x_1)\dots\phi(x_n)\; + \dots\right) \qquad (10.20)
\end{aligned}$$

which can be rewritten as

$$Z[J] = N\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\, J(x_1)\dots J(x_n)\int \mathcal{D}\phi\, e^{i\int d^4x\{\mathcal{L}_0+\mathcal{L}_{\rm int.}\}}\,\phi(x_1)\dots\phi(x_n)\;. \qquad (10.21)$$

On the other hand, equation (10.11) for the n-point correlation function means that we can also write the generating functional as

$$Z[J] = Z[0]\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\, J(x_1)\dots J(x_n)\, G^{(n)}(x_1,\dots,x_n)\;. \qquad (10.22)$$

In fact equations (10.11) and (10.22) are equivalent. By comparing (10.22) with (10.21) we infer that

$$\boxed{G^{(n)}(x_1,\dots,x_n) = \frac{1}{Z[0]}\int \mathcal{D}\phi\, e^{i\int d^4x\{\mathcal{L}_0+\mathcal{L}_{\rm int.}\}}\;\phi(x_1)\dots\phi(x_n)}\;. \qquad (10.23)$$

Equation (10.23) gives as the n-point correlation function in the presence of interactions and directly from a functional integral. We can now implement perturbation theory directly in the correlation functions by expanding the exponential containing $\mathcal{L}_{\rm int.}$ up to the desired order. Let us consider again the two-point function and the four-point function in a real scalar theory in the presence of the interaction (10.8).

Two-point Function:

We want to compute the contributions to the two-point function up to order λ. From (10.23) we have

$$G^{(2)}(x_1,x_2) = \frac{1}{Z[0]}\int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}_0}\,\phi(x_1)\,\phi(x_2)\times\left(1 - i\frac{\lambda}{4!}\int d^4x\,\phi^4(x) + \dots\right)\;, \qquad (10.24)$$

where the dots denote higher orders in λ. If, as we did in the previous section, we take $Z[0] = Z_0[0]$, then it is clear that the functional integrals can be performed using Wick's theorem, since they only depend on the free lagrangian $\mathcal{L}_0$. For instance, the first term is clearly the free propagator $D_F(x_1 - x_2)$, the zeroth order in λ. The second term, the contribution to order λ, is given by

$$-i\frac{\lambda}{4!}\,\frac{1}{Z[0]}\int d^4y\int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}_0}\,\phi(x_1)\,\phi(x_2)\,\phi^4(y)\ . \tag{10.25}$$

Then, using the $Z[0] = Z_0[0]$ approximation, the path integral in (10.25) can be thought of as the vacuum expectation value of the time-ordered product of six fields in the free theory. But Wick theorem tells us how to compute these: they are given by the products of three propagators summed over all the possible pairings. Equation (10.25) can then be rewritten as

$$-i\frac{\lambda}{4!}\int d^4y\ \{3\,D_F(x_1-x_2)\,D_F(y-y)\,D_F(y-y) + 12\,D_F(x_1-y)\,D_F(x_2-y)\,D_F(y-y)\}\ , \tag{10.26}$$

where the factors of 3 and 12 are the combinatoric factors of the two types of diagrams: free propagation from x_1 to x_2 plus vacuum correction, and correction of the propagator to order λ. It can be checked that this result is identical to (10.12) and corresponds to the Feynman diagrams of Figure 10.2. Just as we did then, the bubble diagrams correcting the vacuum will cancel when considering the interactions correcting the $Z[0] = Z_0[0]$ approximation.

Four-point Function:

The four-point function up to order λ is

$$G^{(4)}(x_1,x_2,x_3,x_4) = \frac{1}{Z[0]}\int \mathcal{D}\phi\ e^{i\int d^4x\,\mathcal{L}_0}\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\left(1 - i\frac{\lambda}{4!}\int d^4y\phi^4(y) + \ldots\right)\ . \tag{10.27}$$

Once again, the functional integral is done by using Wick theorem result. For instance, the fully connected diagram of Figure 10.3 is obtained by pairing each external point x_i, with $i = (1-4)$ with each of the points being integrated over. This brings a combinatoric factor of 4! that completely cancels the 4! in the denominator, resulting in the final result of (10.19).

Due to the reliance on Wick theorem, we will refer to the method above as the Wick theorem method. Either way we compute the correlation functions, we can see that there are connected and disconnected diagrams. We will see that all diagrams can be obtained from the connected ones.

10.2 Connected Generating Functional

We have seen that in picturing the results for the correlation functions in perturbation theory in the form of Feynman diagrams, there are connected as well as disconnected diagrams. However, it is clear that all disconnected diagrams can be obtained as the product of connected ones. For instance, consider the case of the diagram on the right in Figure 10.4: it can clearly be the product of the free propagator with the order λ correction to the propagator in the second term of (10.18). In general, if C_I is a connected diagram of type I, we can obtain any diagram D as

$$D = \frac{1}{s_D} \prod_I (C_I)^{n_I} \tag{10.28}$$

where n_I is the number of repeated diagrams of type I and s_D is a symmetry factor that accounts for the exchange among identical connected diagrams. The combinatorics associated with the exchange of identical connected diagrams dictates that

$$s_D = \prod_I n_I! \ . \tag{10.29}$$

The diagram D can be anything, connected of disconnected. But the generating functional is built from the sum of all diagrams. Thus

$$Z[J] \sim \sum_{\{n_I\}} D \ , \tag{10.30}$$

where the sum is over all possible sets of numbers $\{n_I\}$ since we are computing all the diagrams to all order in perturbation theory. Then, we can write

$$Z[J] \sim \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} \ . \tag{10.31}$$

Now to each diagram D corresponds a set of numbers $\{n_I\}$. But since in $Z[J]$ we must include all diagrams to all orders in perturbation theory, the we must be have all possible values of n_I from 0 up to ∞. Thus, we have

$$Z[J] \sim \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} = \prod_I e^{C_I} = e^{\sum_I C_I} \ . \tag{10.32}$$

Our final result is then that the generating functional of all diagrams can be written as an exponential of the sum of al the *connected* diagrams! That is we can define the generating functional for all connected diagrams $iW[J]$ by

$$Z[J] = e^{iW[J]} \ . \tag{10.33}$$

The connected generating functional $iW[J]$ will be useful when we require the connected correlation functions. This will be the case every time we need to compute physical observables, such as transition amplitudes for scattering or the potential energy of a given field configuration.

Additional suggested readings

- *Quantum Field Theory* , by M. Srednicki, Chapter 8.
- *Quantum Field Theory in a Nutshell*, by A. Zee. Chapter 1.7, first few sections on free field theory.