Lecture 1

Why do we need Quantum Field Theory ?

Quantum Field Theory (QFT) is, at least in its origin, the result of trying to work with both quantum mechanics and special relativity. Loosely speaking, the uncertainty principle tells us that we can violate energy conservation by ΔE as long as it is for a small Δt. But on the other hand, special relativity tells us that energy can be converted into matter. So if we get a large energy fluctuation ΔE (for a short Δt) this energy might be large enough to produce new particles, at least for that short period of time. However, quantum mechanics does not allow for such process. For instance, the Schrödinger equation for an electron describes the evolution of just this one electron, independently of how strongly it interacts with a given potential. The same continues to be true of its relativistic counterpart, the Dirac equation. We need a framework that allows for the creation (and annihilation) of quanta. This is QFT.

We can say the same thing by being a bit more precise so that we can start to see how we are going to tackle this problem. Let us consider a *classical* source that emits particles with an amplitude $J_E(x)$, where $x \equiv x_\mu$ is the space-time position. We also consider an absorption source of amplitude $J_A(x)$. We assume that a particle of mass m that is emitted at y propagates freely before being absorbed at x.

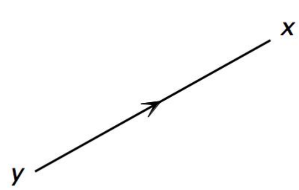

Figure 1.1: Emission, propagation and absorption of a particle.

The quantum mechanical amplitude is given by

$$\mathcal{A} = \int d^4x \, d^4y \, \langle x | e^{-iH\Delta t} | y \rangle \, J_A(x) \, J_E(y) \ , \tag{1.1}$$

where $\Delta t = x_0 - y_0$. Here we have used the notation

$$d^4x \equiv dt \, d^3x \ , \tag{1.2}$$

to denote the Minkowski space four-volume, i.e. we are integrating over time and all space. We want to check if the amplitude in (1.1) is Lorentz invariant, i.e. if it is compatible with special relativity. Writing

$$H = \sqrt{p^2 + m^2} \equiv \omega_p \ , \tag{1.3}$$

as the frequency associated with momentum p, then the amplitude is

$$\mathcal{A} = \int d^4x \, d^4y \, \langle x | e^{-i\omega_p (x_0 - y_o)} | y \rangle \, J_A(x) \, J_E(y) \ . \tag{1.4}$$

If we go to momentum space using

$$|x\rangle = \int \frac{d^3p}{(2\pi)^{3/2}} \, |p\rangle \, e^{-i\vec{p}\cdot\vec{x}} \ , \tag{1.5}$$

and analogously for $|y\rangle$, we obtain

$$\mathcal{A} = \int d^4x \, d^4y \int \frac{d^3p}{(2\pi)^{3/2}} \, \langle p | \, e^{i\vec{p}\cdot\vec{x}} \, e^{-i\omega_p (x_0 - y_o)} \int \frac{d^3p'}{(2\pi)^{3/2}} \, |p'\rangle \, e^{-i\vec{p'}\cdot\vec{y}} \, J_A(x) \, J_E(y) \ . \tag{1.6}$$

Using that

$$\langle p | p' \rangle = \delta^3(\vec{p} - \vec{p'}) \, N_p^2 \ , \tag{1.7}$$

where N_p is the momentum-dependent normalization, we now have

$$\mathcal{A} = \int d^4x \, d^4y \, J_A(x) \, J_E(y) \int \frac{d^3p}{(2\pi)^3} \, N_p^2 \ e^{-ip^\mu (x_\mu - y_\mu)} \ , \tag{1.8}$$

In the last exponential factor in (1.8) we use covariant notation, i.e.

$$p^{\mu}\,(x_{\mu} - y_{\mu}) = p_0(x_0 - y_0) - \vec{p}\cdot(\vec{x} - \vec{y}) = \omega_p \Delta t - \vec{p}\cdot(\vec{x} - \vec{y}) \ . \tag{1.9}$$

To check if $\mathcal{A}$ is Lorentz invariant we are going to define the four-momentum integration with a Lorentz invariant measure. Defining

$$d^4 p = dp_0 \; d^3 p \ , \tag{1.10}$$

we now can compute the Lorentz invariant combination

$$d^4 p \; \delta(p^2 - m^2) \ , \tag{1.11}$$

where the delta function ensures that $p^2 = p_{\mu}\, p^{\mu} = m^2$. Then we do the integral on p_0 as in

$$\int dp_0\, \delta(p^2 - m^2) = \int dp_0\, \delta(p_0^2 - |\vec{p}|^2 - m^2) = \int dp_0\, \frac{\delta(p_0 - \omega_p)}{|2p_0|} = \int dp_0\, \frac{\delta(p_0 - \omega_p)}{2w_p} \ , \tag{1.12}$$

remembering that $\omega_p = +\sqrt{p^2 + m^2}$ positive. Only the positive root contributes in (1.12) since the fact that p^{μ} is always time-like means that the *sign* of p_0 is invariant. This, in turn, means that the p_0 integration interval is $(0, \infty)$, and the negative root is outside the integration region.

This allows us to rewrite the amplitude as

$$\mathcal{A} = \int d^4x\, d^4y\, J_A(x)\, J_E(y) \int \frac{d^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, 2\omega_p \;\, N_p^2 \;\, e^{-ip^{\mu}\,(x_{\mu} - y_{\mu})} \ . \tag{1.13}$$

The expression above appears Lorentz invariant other than for the momentum dependent factor

$$2\omega_p\, N_p^2 \ . \tag{1.14}$$

Thus, the choice (up to an irrelevant constant)

$$N_p^2 = \frac{1}{2\omega_p} \ , \tag{1.15}$$

results in the Lorentz invariant amplitude

$$\mathcal{A} = \int d^4x\, d^4y\, J_A(x)\, J_E(y) \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2 - m^2)\; e^{-ip^\mu\,(x_\mu - y_\mu)}\ . \tag{1.16}$$

Although the quantum mechanical amplitude in (1.16) is manifestly Lorentz invariant, there remains a problem: this expression is valid even if the interval separating x from y is spatial, i.e. even if the separation is non-causal. This is obviously wrong, since we started from the assumption that there is an *emitting* source at y and an *absorbing* source at x, for which the causal order is crucial, which means that the way it is now the separation should not be spatial.

In order to solve this problem,we are going to take a crucial step: we are going to allow *all* sources to both emit and absorb, i.e. at any point x we have

$$J(x) = J_E(x) + J_A(x)\ . \tag{1.17}$$

The amplitude then reads

$$\mathcal{A} = \int d^4x\, d^4y\, J(x)\, J(y) \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, \left\{\theta(x_0 - y_0)\, e^{-ip^\mu\,(x_\mu - y_\mu)} + \theta(y_0 - x_0)\, e^{+ip^\mu\,(x_\mu - y_\mu)}\right\}\ . \tag{1.18}$$

The first term in (1.18) corresponds to the emission in y and absorption in x, since the function $\theta(x_0 - y_0) \neq 0$ for $x_0 > y_0$. For the opposite timel order, this term is zero and then only the second term contributes. The sign inversion in the exponential of the second term in (1.18) needs some explaining. Surely, the time component $p_0\,(y_0 - x_0) = -p_0\,(x_0 - y_0)$ comes from just the inversion of the causal order. However, the inversion of the space component from $\vec{p}\cdot(\vec{x} - \vec{y})$ to $-\vec{p}\cdot(\vec{x} - \vec{y})$ is possible by changing d^3p to $-d^3p$ and swiching the limits of the spatial momentum integration to preserve the overall sign.

So for time like separations, when the order of the events is an observable, only one of these terms contributes. On the other hand, for space like separations *both* terms contribute. Different observers would disagree on the temporal order of the event, however all of them would write the same amplitude. So this amplitude is both Lorentz invariant and causal. It is typically written as

$$\mathcal{A} = \int d^4x\, d^4y\, J(x)\, J(y)\, D_F(x - y)\ , \tag{1.19}$$

where we defined

$$D_F(x - y) \equiv \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, \left\{\theta(x_0 - y_0)\, e^{-ip^\mu\,(x_\mu - y_\mu)} + \theta(y_0 - x_0)\, e^{+ip^\mu\,(x_\mu - y_\mu)}\right\}\ . \tag{1.20}$$

The two-point function above is what is called a Feynman propagator. To summarize so far, in order to obtain a Lorentz invariant and causal quantum mechanical amplitude for the emission, propagation and absorption of a particle we had to allow for all points in spacetime to both emit and absorb, and we needed to allow for all possible time orders. There is still one more thing we need to introduce.

1.1 Charged Particles

Here is the problem: if the particle propagating between y and x is charged, for instance under standard electromagnetism, i.e. electrically charged, then because the amplitude (1.19) does not tell us the order of events in the case of space like separation, we do not know the sign of the current. For instance, suppose a negatively charged particle. Is it being absorbed or emitted ? We concluded above that this absolute statement should not be allowed. But this means that we cannot know the direction of the current.

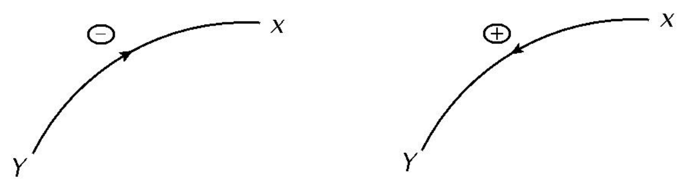

Figure 1.2: Emission, propagation and absorption of a charged particle. Consistence with either temporal order is restored by having anti-particles. Emission of a negatively charged particle at y followed by absorption at x is equivalent to emission of the positively-charged anti-particle at x, followed by absorption at y.

The solution to this problem is that for each negatively charged particle, there must be a positively charged particle with the same mass, its anti-particle. With this addition, it will not be possible to distinguish between say the emission of a negatively charged particle or the absorption of its positively charged anti-particle.

In general, any time a particle has an internal quantum number that may distinguish emission from absorption it should have a distinct anti-particle that would restore the desired indistinguishability. For instance, neutral kaons have no electric charge, but they carry a quantum number called “strangeness” which distinguishes the neutral kaon from the neutral anti-kaon. In the absence of any distinguishing internal quantum number, a particle can be its own anti-particle.

Finally, to illustrate the relationship between propagation and particle or anti-particle identity, we consider the scattering of a particle off a localized potential. We first consider the situation with emission at y, followed by interaction at z and finally absorption at x, i.e. the time order is $x_0 > z_0 > y_0$.

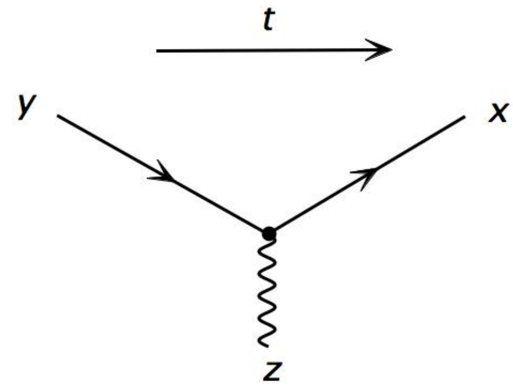

Figure 1.3: Scattering off a localized potential.

The amplitude for this is

$$\mathcal{A}_{\text{scatt.}} = \int d^4x\, d^4y\, J(y)\, D_F(z-y)\, \mathcal{A}_{\text{int.}}(z)\, D_F(z-x)\, J(x)\ , \tag{1.21}$$

where $\mathcal{A}_{\text{int.}}(z)$ is the amplitude for the local interaction with the potential at z. But we know that the amplitude is non-zero even if events are spatially separated. In this case then, it is possible to have a non-zero amplitude corresponding to the following time order: $y_0, x_0 > z_0$. This now would correspond to the diagram in the Figure 1.4.

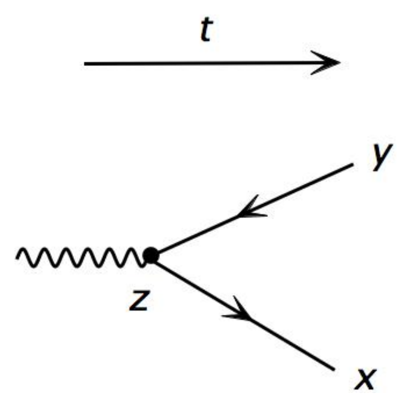

Figure 1.4: Particle – anti-particle pair creation.

In this time order, a pair is created from the "vacuum" at z. The arrows indicate that a particle propagates between z and x, where it is absorbed, whereas an anti-particle travels from z to y. Thus, the creation of a pair particle–anti-particle, assuming there is enough energy, is an unavoidable consequence of the marriage between quantum mechanics and special relativity. All of the arguments above lead us to the fact that relativistic quantum mechanics, compatible with causality, must be a theory of quantized local fields. That is to say, we must be able to create or annihilate quanta of the fields locally, including particles and anti-particles. We will define what we really mean by this in future lectures.

1.2 Description of Condensed Matter Systems

Although the development of QFT historically stems from the combination of quantum mechanics and special relativity, applications to non-relativistic systems are very much possible. This is the case, for instance, of condensed matter systems. The motion of electrons and ions are non-relativistic. However, quantum field theoretic descriptions are of use in many cases. As an example, consider the description of conduction in a solid, where electrons might jump to different bands leaving holes that have transport properties similar to those of a particle with the opposite charge. Thus, having a formalism with particle and anti-particle creation is helpful.

Perhaps more importantly, the microscopic description of condensed matters systems involves a forbidingly large number of particles. However, typically at low energies/large distances the relevant degrees of freedom are simpler as a result of phase transitions. For instance, in an ordered solid phase, there are collective degrees of freedom describing the system at low energies, the phonons. This is fairly common in condensed matter systems. Field theoretic methods are particularly useful to describe the properties of these collective modes of matter. In addition to broken symmetries, there is another way to characterize the long range behavior of a system, related to the topological properties of the fields describing it. Here too the field theory description is extremely useful.

Finally, the concept of renormalization, and in particular the methods of the renormalization group, are central to the understanding of condensed matter systems at different length scales. In this sense, quantum field theory and the renormalization group provide a unifying description of many different systems, both relativistic (as in particle physics or string theory) and non-relativistic. These connections will be made mostly in the second part of the course.

Additional suggested readings

- The discussion above follows *Modern Quantum Field Theory, A Concise Introduction*, by T. Banks, Chapter 1.2.
- See *The Quantum Theory of Fields, Vol.I*, by S. Weinberg, Chapter 1, for a good historical introduction.
- For a good review of special relativity, *Electrodynamics*, by J. D. Jackson, Chapter 11. Particularly important is understanding covariant notation, developed in 11.6 and 11.7.