Quantum Field Theory I – Prof. Gustavo Burdman

Homework 2

Due 08/04/2024

1. **Complex Scalar Field**: (2pts)
Consider a complex scalar field with the action given by:

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right)$$

a) Derive the equations of motion for $\phi(x)$ e $\phi^*(x)$.

b) Find the conjugate momenta of $\phi(x)$ e $\phi^*(x)$, and obtain the Hamiltonian.

c) Expand the fields and their conjugate momenta in momentum space, in the most general way consistent with the equations of motion in terms of creation and annihilation operators. Impose appropriate commutation rules.

d) Write the Hamiltonian H in terms of creation and annihilation operators. Show that there are two types of particles with the same mass m.

2. **Chiral Currents**: (3 pts)

a) Show that the matrix $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$, satisfies

$$(\gamma_5)^2 = 1; \qquad \{\gamma_5, \gamma^\mu\} = 0.$$

In the so-called chiral basis, where we have

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the matrix γ_5 can be written as

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) Using the Dirac equation, show that the fermionic currents

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad\qquad j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma_5\psi \ ,$$

satisfy

$$\partial_\mu j^\mu = 0, \qquad\qquad \partial_\mu j^{\mu 5} = 2\, i\, m\, \bar{\psi}\gamma_5\psi.$$

So in the $m = 0$ limit both currents, the vector and the axial-vector, are conserved.

c) Verify that

$$P_{L,R} \equiv \frac{(1 \mp \gamma_5)}{2}$$

are projection operators. Then, defining $P_{L,R}\psi = \psi_{L,R}$,
show that

$$\begin{aligned} \bar{\psi}\psi &= \bar{\psi}_L\psi_R + h.c. \\ \bar{\psi}\gamma_\mu\psi &= \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R \end{aligned}$$

d) Defining a chiral transformation as $\psi \to \gamma_5\psi$, how do the bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform under it ? Are they invariant ?

3. **Discrete Symmetries and Handedness**: (2 pts)

(a) A parity transformation is defined as a spatial reflection.

$$x^\mu = (x_0, \mathbf{x}) \to x'^\mu = (x_0, -\mathbf{x}) \ .$$

Show that the fermion transformation associated to it, i.e.

$$\psi(x) \to \psi'(x') = S(\Lambda_P)\psi(x) \ ,$$

is given by $S(\Lambda_P) = \eta\, \gamma_0$, where η satisfies $|\eta|^2 = 1$.

(b) The weak interactions can be described by the so-called Fermi lagrangian given by

$$\mathcal{L} = G\, (\bar{\psi}_{1L}\gamma^\mu\psi_{2L})\, (\bar{\psi}_{3L}\gamma_\mu\psi_{4L}) \ ,$$

where ψ_{iL} with $i = 1, 2, 3, 4$ are left-handed fermions (in principle of different flavors i) and G is a constant. Show that this lagrangian violates parity invariance.

(c) Show that if ψ is a left-handed fermion, then the conjugate field, defined as $\psi_c = C\gamma^0\psi^*$, is a right-handed fermion.

4. **Majorana Fermions** (3 pts):

(a) The Dirac equation implies that $i \not{\partial}\psi = m\psi$. Show that the equations

$$i \not{\partial}\psi = m\psi_c$$

and

$$i \not{\partial}\psi_c = m\psi \ ,$$

where $\psi_c = C\gamma^0\psi^*$ is the conjugate fermion field (e.g. $C = i\gamma_2\gamma_0$ in the Dirac representation), are compatible with the relativistic dispersion condition, i.e. prove that their use leads to the Klein-Gordon equation for ψ (and also for ψ_c).

(b) Show that:

$$\bar{\psi}_c\psi = \psi^T C\psi \qquad \bar{\psi}\psi_c = -\psi^{*T}C\psi^*$$

$$(\bar{\psi}_c\psi)^\dagger = \bar{\psi}\psi_c \qquad \bar{\psi}_{R,c}\psi_L = 0$$

(c) Show that

$$\psi^T C\psi$$

in invariant under Lorentz transformation. Then argue why this allows us to write mass terms as

$$\begin{aligned}\mathcal{L}_{\text{Majorana mass}} &= -\frac{1}{2}m\left\{\psi^T\,C\,\psi + -\psi^*\,C\,\psi^{*T}\right\} \\ &= -\frac{1}{2}m\left\{\bar{\psi}_c\psi + \bar{\psi}\psi_c\right\}\end{aligned}$$

(d) Show that the mass terms above violate fermion number by two units.

(e) Show that, unlike for the case of the Dirac mass term, the mass terms above *preserve* handendness, i.e. they are of the form LL and RR.