Quantum Field Theory I - Prof. Gustavo Burdman

Homework 1

Due 20/03/2024

1. **Relativity Gymnastics**

   (a) Show that the invariance of the interval between two events $(t^2 - |\vec{x}|^2)$ under Lorentz transformations is satisfied if we define the position four-vector $x^\mu \equiv (t; \vec{x})$ such that

   $$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu}$$

   where we defined the Lorentz transformations as Λ^{μ}_{ν}. The inverse Lorentz transformation is defined by $x^\nu = \Lambda^{\nu}_{\mu}\, x'^{\mu}$, and it satisfies

   $$\Lambda^{\mu}_{\nu}\, \Lambda^{\nu}_{\rho} = \delta^{\mu}_{\rho}$$

   (b) The Minkowski space metric is given by

   $$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

   i. A contra-variant four-vector is denoted as $v^\mu = (t; \vec{v})$ Show that if we define a covariant four-vector as $v_\mu \equiv (t; -\vec{v})$, we can write $v.v = v^\mu v_\mu$.

   ii. Verify that $v_\mu = g_{\mu\nu}\, v^\nu$, and $v^\mu = g^{\mu\nu}\, v_\nu$.

   iii. Verify that the interval is given by the position four-vector squared defined by $x.x \equiv x^\mu g_{\mu\nu} x^\nu$.

   (c) Show that the relativistic dispersion relation $E^2 - |\vec{p}|^2 = m^2$, where m is the particle mass, is consistent with defining the momentum four-vector as $P^\mu = (E; \vec{p})$.

(d) Define the differential operators

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu \equiv (\frac{\partial}{\partial t}; \vec{\nabla}) \qquad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu \equiv (\frac{\partial}{\partial t}; -\vec{\nabla})$$

i. Construct the Klein-Gordon operator using the identifications

$$E \to i\frac{\partial}{\partial t} \qquad \vec{p} \to -i\vec{\nabla}$$

ii. Show that if we define the four-vector current $J^\mu \equiv (\rho; \vec{j})$, where ρ and $\vec{j}$ the charge density and current respectively, the continuity equation in electrodynamics can be written as the conservation of the four-current as

$$\partial_\mu J^\mu = 0$$

2. **About those Delta Functions**
Using the defining property of the Delta function

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$

for some function $f(x)$, and that $\delta(-x) = \delta(x)$

(a) Show that

$$\int_{-\infty}^{+\infty} f(x)\delta(a\,x)\,dx = \frac{1}{|a|}\,f(0)$$

for an arbitrary constant a.

(b) Using all these show that for a function $g(x)$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx = \sum_a \frac{1}{|g'(a)|}\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx\ ,$$

where a are the zeroes of $g(x)$ (i.e. $g(a) = 0, \forall a$) while $g'(a) \neq 0$.

(c) Now verify that for $\omega_p = \sqrt{(\vec{p})^2 + m^2} > 0$ we have

$$\int d^4p\,\delta(P^2 - m^2)\,f(P) = \int d^3p \int dp_0\,\delta(p_0^2 - (\vec{p})^2 - m^2)\,f(P) = \frac{1}{2\omega_p}\int d^3p\,f(\omega_p, \vec{p})\ ,$$

where you need to use the fact that since the four-momentum is always time-like, the sign of p_0 is Lorentz invariant. Thus we need only integrate p_0 from 0 to ∞, which in turn means we only need the positive root.

3. **Electromagnetism**:

The covariant form of the Lagrangian density for the electromagnetic field in the absence of sources is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor defined in terms of the potential 4-vector A_μ.

(a) Derive the equations of motion for $A_\mu(x)$ assuming it is the dynamical field, using Euler-Lagrange. Show that these are equivalent to Maxwell's equations in the vacuum. (It is useful to remember that $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.)

(b) If we now consider sources in the form of a current j_μ, what is the form of the interaction term between A_μ and j_μ we must add to $\mathcal{L}$ in order to obtain Maxwell's equations in the presence of the source $j_\mu = (\rho, \vec{j})$?

(c) The Lagrangian in the absence of sources is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \ ,$$

with $\alpha(x)$ an arbitrary function. What is the condition that j_μ must fulfill for the Lagrangian to remain gauge invariant in the presence of sources ?